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AIRCRAFT COMPENSATOR DESIGN METHODS.(U)

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AIRCRAFT COMPENSATOR DESIGN METHODS

Tom L. Riggs, Jr.

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AIRCRAFT COMPENSATOR DESIGN METHODS

Tom L. Riggs, Jr.

A Thesis
Submitted to
the Graduate Faculty of
Auburn University
in Partial Fulfillment of the
Requirements for the
Degree of
Master of Electrical Engineering

Auburn, Alabama
June 7, 1977

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AIRCRAFT COMPENSATOR DESIGN METHODS

Tom L. Riggs, Jr.

Tom L. Riggs, Jr.
Signature of Author

25 March 1977
Date

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VITA

Tom Lance Riggs, Jr., son of Tom Lance and Fern Marie (Lowry) Riggs, was born in Washington, D. C. on April 21, 1947. He attended public school in Washington, D. C. until the age of 15 when he and his family moved to Arlington, Virginia. He completed his secondary education in Arlington County Public Schools. In April, 1966, he enlisted in the U. S. Air Force. He served in the U. S. Air Force as an electronics technician until 1973 whereupon he was selected by the U. S. Air Force to attend Auburn University to attain a baccalaureate degree in Electrical Engineering with the ultimate goal of being commissioned a 2nd Lieutenant in the U. S. Air Force. He entered Auburn University in September, 1973 and received the degree of Bachelor of Electrical Engineering in March, 1976. By delaying his commission for one year, the U. S. Air Force selected him to enter the Graduate School, Auburn University, in March, 1976. He married Patricia Frances, daughter of John William and Margaret May (Donnison) Davis of Manchester, England in December, 1969. They have three children, Dawn Michelle, Karen Nichole, and Kimberly Annette.

THESIS ABSTRACT

AIRCRAFT COMPENSATOR DESIGN METHODS

Tom L. Riggs, Jr.

Master of Electrical Engineering, June 7, 1977
(B.E.E., Auburn University, 1976)

132 Typed Pages

Directed by Joseph S. Boland, III

In this paper methods using linear analysis are presented for the design of longitudinal flight compensators. The compensators employ state feedback to force the aircraft to respond in the desired manner. The design process involves analysis of the uncompensated aircraft, construction of a model which has the desired response, and two algorithms for designing the compensator.

A method is presented for contriving the desired model from handling qualities performance criteria. The method is straightforward and results in the exact desired short period response; however, the phugoid (long period) response is less predictable.

The compensator design algorithms are easily implemented into computer programs. The algorithms require a minimum of human/computer interaction and solutions are assured for controllable systems.

Compensators for two aircraft are designed to show the techniques presented. In both examples the compensated system response is exactly the same as the model response.

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I. INTRODUCTION

The modern aircraft is capable of flight over a wide range of velocities and altitudes. This increased capability has resulted in a deterioration in airplane stability manifested by an increase in the airframe natural frequencies and a decrease in the airframe damping [1]. In order to improve the stability of the modern aircraft, automatic control devices have been devised. These control devices, often called Stability Augmentation Systems (SAS), operate almost universally by sensing the airframe motions and then moving a control surface to oppose the airframe motion.

The problem of designing an SAS is complex. The equations which represent the mathematical model of the airframe are nonlinear differential equations. For reasons of comfort, safety and mission performance, it is desirable that the airplane fly along a smooth path. It is reasonable to consider departures from the path as small perturbations. This assumption permits considerable simplification of the airframe equations of motion by reducing the model to two independent linear differential systems, one representing longitudinal flight and the other representing lateral flight.

The design problem has now been broken down into two independent problems, one for longitudinal flight and the other for lateral flight. Although the problem has been simplified greatly, the design of the SAS for the two flight configurations is by no means trivial. For

instance, the small disturbance mathematical model for longitudinal flight is a 5th order linear differential system with multiple inputs and outputs. In order to design an SAS for longitudinal flight, the system open loop transfer function and the desired transfer function must be known. Then using some algorithm the compensator is designed such that the compensated system transfer function is the same as the desired transfer function.

One method for designing the SAS is by determining the transfer function for a particular state with respect to some input and designing a compensator that places the poles and zeroes of the transfer function at some desired location. However this method is impractical for even moderate size systems because of coupling between states.

A more practical method is to generate a model, normally of the same dimension as the plant and often of the same structure as the plant, that has the overall desired response for the system. The problem now reduces to determining a compensator that will force the states of the plant to equal the states of the model. The bulk of the work in attempting to solve this problem has used optimal control theory. To do this requires the generation of a cost function. Most papers written in this area specify a quadratic performance measure as given in Equation (1-1).

$$J = \int_0^\infty [(x_p - x_m)^T Q(x_p - x_m) + u^T Ru]dt \qquad (1-1)$$

where

x_p = states of the plant

x_m = states of the model

u = control applied

and Q and R are nxn and mxm weighting matrices respectively. This problem has been solved by Ryanski, Reynolds, and Shed [2] and Windsor and Roy [3] among others. However the key difficulty in all of these solutions is the lack of definition of the matrices Q and R.

R. T. Curran [4] developed an algorithmic approach to the design of the compensator. By performing transformations on the plant and model systems he transforms the system to a special canonical form which allows him to easily determine the compensator which equates the transfer functions of the model and compensated plant. Since the transfer function for a given system is not dependent on the coordinate system of the state equations, a solution to the problem is assured. Once the compensator is determined it is transformed back to the original coordinate system.

The aircraft control problem lends itself to Curran's method because of the form of the aircraft system matrices. In this paper, Curran's method is fully developed in relation to the aircraft longitudinal control problem. Curran's algorithm is simplified and extended to include a wider range of systems. This paper shows that a solution to the aircraft longitudinal control problem via Curran's method is

always assured if care is taken in constructing the model. Also a method is developed for generating the state equations for the model based on desired performance criteria [5].

II. DEVELOPMENT OF AIRCRAFT LONGITUDINAL EQUATIONS AND PERFORMANCE CRITERIA

Aerodynamic forces are, in general, roughly proportional to $\rho V^2 \ell^2$ where ρ is the density of air, V is the velocity relative to the local environment, and ℓ is a characteristic linear dimension. It is therefore universal practice to obtain nondimensional force coefficients by dividing the aerodynamic forces by a factor proportional to the above quantity. For example, the weight coefficient is given by $C_W = 2mg/\rho V^2 S$ where mg is the weight of the aircraft and S is usually the wing surface area. Table 2-1 lists the aerodynamic forces related to longitudinal flight, the small disturbance divisor and the resultant nondimensional quantity [6]. The constants in Table 2-1 are defined below.

V_e = constant reference velocity

ρ_e = constant reference density

$\bar{c}$ = longitudinal reference length (wing mean chord)

S = reference wing plan area

Using the nondimensional quantities of Table 2-1, a set of 1st order linear nondimensional dynamic equations can be written which will describe the aircraft's longitudinal motion. The equations were taken from Etkin's Dynamics of Atmospheric Flight [6] and are given by Equations (2-1) through (2-5).

Table 2-1. Dimensional and Nondimensional Aerodynamic Forces.

Dimensional Quantity	Symbol	Small Disturbance Divisor	Nondimensional Quantity
Thrust, Drag, Lift	T, D, L	$\frac{1}{2}\rho V^2 S$	C_T, C_D, C_L
Weight	mg	$\frac{1}{2}\rho V^2 S$	C_W
Moment	M	$\frac{1}{2}\rho V^2 S\bar{c}$	C_m
Pitch Rate	q	$2V_e/\bar{c}$	$\hat{q}$
Mass	m	$\frac{1}{2}\rho_e S\bar{c}$	μ
Inertia	I_y	$\rho_e S(\bar{c}/2)^3$	$\hat{I}_y$
Density	ρ	ρ_e	$\hat{\rho}$
Velocity	V	V_e	$\hat{V}$
Altitude	Z	$\bar{c}/2$	$\hat{Z}$

$$
\begin{aligned}
2\mu(D\hat{V}) = {} & [C_{T_V}\cos(\alpha_T) - C_{D_V} + 2C_{W_e}\sin(\gamma_e)]\Delta\hat{V} \\
& - [C_{D_\alpha} + C_{T_e}\sin(\alpha_T)]\Delta\alpha - [C_{W_e}\cos\gamma_e]\Delta\gamma \\
& + [C_{T_Z}\cos(\alpha_T) - C_{D_Z} + \frac{\partial\hat{\rho}}{\partial\hat{Z}}\, C_{W_e}\sin(\gamma_e)]\Delta\hat{Z} \\
& + \Delta C_{T_c}\cos(\alpha_T) - \Delta C_{D_c}
\end{aligned}
\qquad (2\text{-}1)
$$

$$
\begin{aligned}
-(2\mu + C_{L_{\dot{\alpha}}})(D\alpha) = {} & [C_{T_V}\sin(\alpha_T) + C_{L_V} + 2C_{W_e}\cos(\gamma_e)]\Delta\hat{V} \\
& + [C_{L_\alpha} + C_{T_e}\cos(\alpha_T)]\Delta\alpha + [C_{L_q} - 2\mu]\hat{q} \\
& + [C_{W_e}\sin(\gamma_e)]\Delta\gamma \\
& + [C_{L_Z} + C_{T_Z}\sin(\alpha_T) + \frac{\partial\hat{\rho}}{\partial\hat{Z}}\, C_{W_e}\cos(\gamma_e)]\Delta\hat{Z} \\
& + \Delta C_{T_c}\sin(\alpha_T) + \Delta C_{L_c}
\end{aligned}
\qquad (2\text{-}2)
$$

$$\hat{I}_y D\hat{q} - C_{m_{\dot{\alpha}}} D\alpha = C_{m_V}\Delta\hat{V} + C_{m_\alpha}\Delta\alpha + C_{m_q}\hat{q} + C_{m_Z}\Delta\hat{Z} + \Delta C_{m_c} \qquad (2\text{-}3)$$

$$D\alpha + D\gamma = \hat{q} \qquad (2\text{-}4)$$

$$D\hat{Z} = -\sin(\gamma_e)\Delta\hat{V} - \cos(\gamma_e)\Delta\gamma - \sin\gamma_e \qquad (2\text{-}5)$$

The symbols in the Equations (2-1) through (2-5) are defined below.

e = reference state

$$D\varepsilon = \frac{\bar{c}}{2V_e}\frac{d\varepsilon}{dt} = \frac{\bar{c}}{2V_e}\dot{\varepsilon} \qquad (2\text{-}6)$$

$$C_{a_b} = \left.\frac{\partial C_a}{\partial b}\right|_e$$

ΔC_{ε_c} = nondimensionalized command input ε

α_T = angle between the thrust vector and the longitudinal body axis

γ_e = steady state angle of climb

Often to simplify the equations, a new angle, θ, is defined such that

$$\theta = \alpha + \gamma \tag{2-7}$$

Figure 2-1 illustrates the angles of longitudinal flight. For simulation purposes it is assumed that at time t=0 the aircraft is in equilibrium and is flying straight and level. Equilibrium of the reference state connects the thrust, drag, lift, and weight coefficients by the following relations

$$C_{T_e}\cos\alpha_T - C_{D_e} = C_{W_e}\sin\gamma_e \tag{2-8}$$

$$C_{T_e}\sin\alpha_T + C_{L_e} = C_{W_e}\cos\gamma_e \tag{2-9}$$

Level flight is defined as

$$\gamma_e = 0 \tag{2-10}$$

If the thrust, drag, lift, and moment all vary with the air density when the speed is constant then, it is a good assumption [6] that

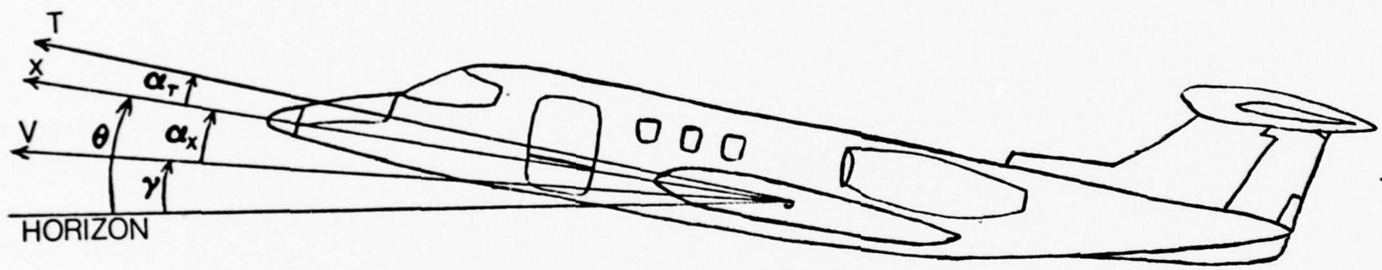


Figure 2-1. Angles of Longitudinal Flight.

$$C_{T_z} = C_{D_z} = C_{L_z} = C_{m_z} = 0 \tag{2-11}$$

Substituting Equations (2-6) through (2-11) and the definitions for $\hat{V}$, $\hat{q}$, and $\hat{Z}$ into Equations (2-1) through (2-5), the longitudinal equations become

$$\Delta\dot{V} = \frac{V_e}{\mu\bar{c}}(C_{T_V}\cos\alpha_T - C_{D_V})\Delta V + \frac{V_e^2}{\mu\bar{c}}(C_{L_e} - C_{D_\alpha})\Delta\alpha$$

$$+ \frac{V_e^2}{\mu\bar{c}}(-C_{W_e})\theta + \frac{V_e^2}{\mu\bar{c}}(\Delta C_{T_c}\cos\alpha_T - \Delta C_{D_c}) \tag{2-12}$$

$$\Delta\dot{\alpha} = -\frac{2}{\bar{c}}\left(\frac{C_{T_V}\sin\alpha_T + C_{L_V} + 2C_{W_e}}{2\mu + C_{L_{\dot{\alpha}}}}\right)\Delta V - \frac{2V_e}{\bar{c}}\left(\frac{C_{L_\alpha} + C_{D_e}}{2\mu + C_{L_{\dot{\alpha}}}}\right)\Delta\alpha$$

$$+ \left[\frac{2\mu - C_{L_q}}{2\mu + C_{L_{\dot{\alpha}}}}\right] q - \left(\frac{4V_e}{\bar{c}^2}\right)\left(\frac{\partial\hat{\rho}}{\partial\hat{z}}\right)\left[\frac{C_{W_e}}{2\mu + C_{L_{\dot{\alpha}}}}\right]\Delta Z$$

$$- \frac{2V_e}{\bar{c}}\left[\frac{\Delta C_{T_c}\sin\alpha_T + \Delta C_{L_c}}{2\mu + C_{L_{\dot{\alpha}}}}\right] \tag{2-13}$$

$$\dot{\theta} = q \tag{2-14}$$

$$\dot{q} = \left[\frac{2}{\bar{c}}\right]^2 \frac{1}{\hat{I}_y}\left[C_{m_V} - C_{m_{\dot{\alpha}}}\frac{(C_{T_V}\sin\alpha_T + C_{L_V} + 2C_{W_e})}{2\mu + C_{L_{\dot{\alpha}}}}\right]\Delta V$$

$$+ \left[\frac{2V_e}{\bar{c}}\right]^2 \frac{1}{\hat{I}_y}\left[C_{m_\alpha} - \frac{C_{m_{\dot{\alpha}}}(C_{L_\alpha} + C_{D_e})}{2\mu + C_{L_{\dot{\alpha}}}}\right]\Delta\alpha$$

$$+ \frac{2V_e}{\bar{c}} \frac{1}{\hat{I}_y} \left[C_{m_q} + \frac{C_{m_{\dot{\alpha}}}(2\mu - C_{L_q})}{2\mu + C_{L_{\dot{\alpha}}}} \right] q$$

$$+ \frac{8V_e^2}{\bar{c}^3} \frac{1}{\hat{I}_y} \left[\frac{C_{m_{\dot{\alpha}}} \frac{\partial \hat{\rho}}{\partial \hat{z}} C_{w_e}}{2\mu + C_{L_{\dot{\alpha}}}} \right] \Delta Z$$

$$+ \frac{2V_e}{\bar{c}}^2 \frac{1}{\hat{I}_y} \left[\Delta C_{m_c} - C_{m_{\dot{\alpha}}} \frac{(\Delta C_{T_c} \sin\alpha_T + \Delta C_{L_c})}{2\mu + C_{L_{\dot{\alpha}}}} \right] \quad (2\text{-}15)$$

$$\Delta\dot{Z} = V_e \Delta\alpha - V_e \theta \quad (2\text{-}16)$$

Equations (2-11) *through* (2-16) *can now be written in state space form.*

The equations are given by

$$\underline{\dot{x}} = A\underline{x} + B\underline{u} \quad (2\text{-}17)$$

where

$$\underline{x} = \begin{bmatrix} \Delta V \\ \Delta\alpha \\ \theta \\ q \\ \Delta Z \end{bmatrix} \quad \text{and} \quad \underline{u} = \begin{bmatrix} \Delta C_{T_c} \\ \Delta C_{L_c} \\ \Delta C_{m_c} \\ \Delta C_{D_c} \end{bmatrix}$$

A and B are matrices of dimension 5x5 and 5x4 respectively and are *shown in Figures* 2-2 *and* 2-3.

$\frac{V_e}{\mu\bar{c}}\left(C_{T_v}\cos\alpha - C_{D_v}\right)$	$-\frac{V_e^2}{\mu\bar{c}}\left(C_{D_\alpha} - C_{L_e}\right)$	$-\frac{V_e^2}{\mu\bar{c}} C_{w_e}$	0	0
$-\frac{2}{\bar{c}}\frac{C_{T_v}\sin\alpha_T + C_{L_v} + 2C_{w_e}}{2\mu + C_{L_{\dot{\alpha}}}}$	$-\frac{2V_e}{\bar{c}}\left[\frac{C_{L_\alpha} + C_{D_e}}{2\mu + C_{L_{\dot{\alpha}}}}\right]$	0	$\frac{2\mu - C_{L_q}}{2\mu + C_{L_{\dot{\alpha}}}}$	$-\frac{4V_e}{\bar{c}^2}\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{C_{w_e}}{2\mu + C_{L_{\dot{\alpha}}}}$
0	0	0	1	0
$\frac{2}{\hat{I}_y\bar{c}}\left[\frac{2C_{m_v}}{\bar{c}} + C_{m_{\dot{\alpha}}}(a_{21})\right]$	$\frac{2V_e}{\hat{I}_y\bar{c}}\left[\frac{2V_e C_{m_\alpha}}{\bar{c}} + C_{m_{\dot{\alpha}}}(a_{22})\right]$	0	$\frac{2V_e}{\hat{I}_y\bar{c}}\left[C_{m_q} + C_{m_{\dot{\alpha}}}(a_{24})\right]$	$-\frac{2V_e}{\hat{I}_y\bar{c}}\left[C_{m_{\dot{\alpha}}}(a_{25})\right]$
0	V_e	$-V_e$	0	0

Figure 2-2. The A-Matrix for the Aerodynamic Equations of Longitudinal Flight in State Space Form.

$\frac{V_e^2}{u\bar{c}} \cos\alpha_T$	0	0	$-\frac{V_e^2}{u\bar{c}}$
$-\frac{2V_e \sin\alpha_T}{\bar{c}(2\mu+c_{L_{\dot{\alpha}}})}$	$-\frac{2V_e}{\bar{c}(2\mu+C_{L_{\dot{\alpha}}})}$	0	0
0	0	0	0
$-\left[\frac{2V_e}{\bar{c}}\right]^2 \frac{1}{\hat{I}_y} \frac{C_{m_{\dot{\alpha}}} \sin\alpha_T}{2\mu+C_{L_{\dot{\alpha}}}}$	$-\left[\frac{2V_e}{\bar{c}}\right]^2 \frac{1}{\hat{I}_y} \frac{1}{2\mu+C_{L_{\dot{\alpha}}}}$	$\left[\frac{2V_e}{\bar{c}}\right]^2 \frac{1}{\hat{I}_y}$	0
0	0	0	0

Figure 2-3. The B-Matrix for the Aerodynamic Equations of Longitudinal Flight in State Space Form.

In the longitudinal flight configuration for an aircraft, there are two distinct characteristic response modes. The first is a short period mode which usually has a natural frequency of approximately 3 radians/sec in comparison to the second, called the phugoid mode, which has a rather long period (usually between 50 and 150 seconds). These response modes are, of course, related to the eigenvalues of the system which in turn are dependent on the aerodynamic coefficients of the aircraft.

The eigenvalues are solutions to the characteristic equation which is given by

$$\text{C.E.} = |\lambda I - A| = 0 \tag{2-18}$$

where

$\lambda \triangleq$ eigenvalue

$I \triangleq$ *Identity matrix*

A = A matrix in Equation (2-17)

The characteristic equation is an n^{th} order polynomial where n is the dimension of the A matrix.

For the fifth order aircraft model there will be five eigenvalues, which may be real or complex. There are three possibilities that must be examined.

1. All eigenvalues are real.
2. Three eigenvalues are real and two are complex conjugates.
3. One eigenvalue is real and four are complex (two pairs of complex conjugates).

The third case is by far the most common with aircraft although 2 and 1 do occur. If the eigenvalues for the system fit the third case, the characteristic equation can be written

$$(\lambda + c_1)(\lambda^2 + b_2\lambda + c_2)(\lambda^2 + b_3\lambda + c_3)=0 \tag{2-19}$$

The second order factors of the characteristic equation can be represented by

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \tag{2-20}$$

where

ζ = damping ratio

ω_n = undamped natural frequency

One second order factor determines the response of the short period mode and the other provides the phugoid mode response.

An inspection of the generalized A matrix (Figure 2-2) reveals that the elements vary under different flight conditions, e.g. velocity and altitude. Simply knowing the aerodynamic coefficients for a given aircraft under given flight conditions and being able to plug in the numbers to compute the elements of the A matrix to calculate the eigenvalues for one situation leaves much to be desired with respect to the general aircraft control problem. In the design of the compensator it is important to know which elements of the A matrix dominate the response modes of the system.

In order to analyze the aircraft longitudinal response, schemes based on empirical results have been devised to approximate the short

period and phugoid modes. Since the z derivatives are small, it is almost universal practice to neglect the ΔZ state. Experimental results reveal the short period oscillation occurs at virtually constant speed (therefore $\Delta V \approx 0$ in the short period mode) while the phugoid takes place at virtually constant angle of attack (therefore $\Delta\alpha \approx 0$ in the phugoid mode) [6].

Short Period Approximation

The short period approximation neglects the ΔV and ΔZ states, reducing the model to a third order system. Equation (2-21) gives the reduced homogeneous equations in state variable form.

$$\begin{bmatrix} D\alpha \\ D\theta \\ D\hat{q} \end{bmatrix} = \begin{bmatrix} -\dfrac{(C_{L_\alpha} + C_{D_e})}{2\mu + C_{L_{\dot{\alpha}}}} & 0 & \dfrac{2\mu - C_{L_q}}{2\mu + C_{L_{\dot{\alpha}}}} \\ 0 & 0 & 1 \\ \dfrac{1}{\hat{I}_y}\left[C_{m_\alpha} - \dfrac{C_{m_{\dot{\alpha}}}(C_{L_\alpha} + C_{D_e})}{2\mu + C_{L_{\dot{\alpha}}}}\right] & 0 & \dfrac{1}{\hat{I}_y}\left[C_{m_q} + \dfrac{C_{m_{\dot{\alpha}}}(2\mu - C_{L_q})}{2\mu + C_{L_{\dot{\alpha}}}}\right] \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \theta \\ \hat{q} \end{bmatrix} \tag{2-21}$$

Clearly, the reduced matrix is singular and has rank = 2, therefore one of its eigenvalues is equal to zero and of little interest. From this approximation it is clear that the values of the a_{22}, a_{24}, a_{42}, and a_{44} elements of the A matrix determine the short period response. Therefore the equations which determine the short period response are

$$\begin{bmatrix} D\alpha \\ D\hat{q} \end{bmatrix} = \begin{bmatrix} -\dfrac{(C_{L_\alpha} + C_{D_e})}{2\mu + C_{L_{\dot{\alpha}}}} & \dfrac{2\mu - C_{L_q}}{2\mu + C_{L_{\dot{\alpha}}}} \\ \dfrac{1}{\hat{I}_y}\left[C_{m_\alpha} - \dfrac{C_{m_{\dot{\alpha}}}(C_{L_\alpha} + C_{D_e})}{2\mu + C_{L_{\dot{\alpha}}}}\right] & \dfrac{1}{\hat{I}_y}\left[C_{m_q} + \dfrac{C_{m_{\dot{\alpha}}}(2\mu - C_{L_q})}{(2\mu + C_{L_{\dot{\alpha}}})}\right] \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \hat{q} \end{bmatrix} \tag{2-22}$$

The Phugoid Approximation

The phugoid approximation is not as clear-cut as the short period approximation. Unfortunately, one cannot break the A matrix down into a submatrix that will give obvious results as with the short period approximation. Therefore it is not possible to calculate the phugoid component of the characteristic equation directly from any given elements of the A matrix. However, the phugoid can be predicted fairly accurately from the aerodynamic coefficients of the aircraft.

Recalling that in the phugoid mode the angle of attack, α, is virtually a constant, hence, $\Delta\alpha=0$. This implies zero pitching moment, so that pitch equilibrium is always maintained. This suggests that the pitching moment equation should be dropped. The reduced equations are given by Equation (2-23).

$$\begin{bmatrix} D\hat{V} \\ 0 \\ D\theta \end{bmatrix} = \begin{bmatrix} \dfrac{C_{T_V} - C_{D_V}}{2\mu} & 0 & -\dfrac{C_{W_e}}{2\mu} \\ -\dfrac{C_{L_V} + 2C_{W_e}}{2\mu + C_{L_{\dot{\alpha}}}} & \dfrac{2\mu - C_{L_q}}{2\mu + C_{L_{\dot{\alpha}}}} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\hat{V} \\ \hat{q} \\ \Delta\theta \end{bmatrix} \tag{2-23}$$

Since usually,

$$2\mu >> C_{L_{\dot{\alpha}}}$$

$$2C_{w_e} >> C_{L_V}$$

$$2\mu >> C_{L_q}$$

$$C_{T_V} >> C_{D_V}$$

Then

$C_{L_{\dot{\alpha}}}$, C_{L_V}, C_{L_q}, and C_{D_V} will be neglected.

The second of the three equations is an algebraic relation, i.e. with the preceding approximations

$$-\frac{C_{w_e}}{\mu}\Delta\hat{V} + \hat{q} = 0 \tag{2-24}$$

Using the approximations and Equation (2-24) to eliminate $\hat{q}$, Equation (2-23) can be shown to be

$$\begin{bmatrix} D\hat{V} \\ D\theta \end{bmatrix} = \begin{bmatrix} \dfrac{C_{T_V}}{2\mu} & -\dfrac{C_{w_e}}{2\mu} \\ \dfrac{C_{w_e}}{\mu} & 0 \end{bmatrix} \begin{bmatrix} \Delta\hat{V} \\ \Delta\theta \end{bmatrix} \tag{2-25}$$

Recalling that

$$\Delta\hat{V} = \frac{\Delta V}{V_e}$$

$$D\hat{V} = \frac{\bar{c}}{2V_e}\frac{d\hat{V}}{dt}$$

$$D\theta = \frac{\bar{c}}{2V_e}\frac{d\theta}{dt}$$

Equation (2-25) can be written

$$\begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \dfrac{V_e C_{T_V}}{\mu\bar{c}} & -\dfrac{V_e^2 C_{W_e}}{\mu\bar{c}} \\ \dfrac{2C_{W_e}}{\mu\bar{c}} & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\theta \end{bmatrix} \tag{2-26}$$

From Equation (2-26) ω_{n_p} and ζ_p can be determined.

$$\omega_{n_p} = \frac{\sqrt{2}\, V_e C_{W_e}}{\mu\bar{c}} \tag{2-27}$$

$$\zeta_p = -\frac{1}{2\sqrt{2}}\frac{C_{T_V}}{C_{W_e}} \tag{2-28}$$

With the knowledge of how ζ_p and ω_{n_p} vary with the aerodynamic coefficients, the elements of the A matrix which are dominantly dependent on these coefficients can be determined by inspection. An examination of the A matrix reveals a_{13} and a_{21} are dominated by C_{W_e} and μ;

therefore they effect ω_{n_p}. The a_{11} element is dominated by C_{T_V} and μ; therefore it effects ζ_p.

Knowledge of which elements of the A matrix effect the short period and phugoid response modes enables one to construct a model which will respond to some desired performance criteria. Because of the indefiniteness of the dominate phugoid elements, a trial and error technique will be required to get the exact desired phugoid response.

Performance Criteria

Much research has been done in the area of aircraft handling qualities to ascertain which vehicle parameters influence pilot acceptance. In investigating the handling qualities related to longitudinal dynamics the problem is usually separated into two parts, associated with the short period response and the phugoid response. Attempts are then made to correlate pilot opinion with various parameters.

First consider the phugoid response. For conventional fixed wing airplanes the phugoid period is very long and not a significant factor in pilot rating, however the phugoid damping is important [6]. As the damping decreases, the pilot must devote more attention to controlling the low frequency motion. One study by F. O'Hara using simulators showed that a damping ratio of .2 was rated acceptable and one of .6 was rated good by the pilots participating in the study [6].

The parameters that effect the pilot ratings of the short period response are more complex than the phugoid response. Substantial disagreement among results based on simply correlating pilot ratings

with short period damping and natural frequency has resulted in a search for more meaningful parameters. One such parameter was derived by noting that the pilot's opinion of an aircraft's handling qualities is influenced by the vehicle's response to control inputs. This depends on both the poles and zeroes of the system transfer function. An important transfer function is the approximate one relating pitch rate response to elevator angle input [6]. This transfer function in the s domain is given by Equation (2-29)

$$\frac{q(s)}{\Delta\delta_e(s)} = \frac{M_\delta\left(s + \frac{L_\alpha}{mV}\right)}{I_y\left(s^2 + 2\zeta_{sp}\omega_{n_{sp}} s + \omega_{n_{sp}}^2\right)} \tag{2-29}$$

To obtain the impulse response of Equation (2-29) let $\Delta\delta_e(s) = 1$ and take the inverse Laplace transform as given by Equation (2-30)

$$q(t) = \frac{M_\delta \varepsilon^{-\zeta\omega_n t}}{I_y\sqrt{1-\zeta^2}}\left[\frac{\tau^2}{\omega_n^2} - \frac{2\zeta\tau}{\omega_n} + 1\right]^{\frac{1}{2}} \sin(\omega t - \phi) \tag{2-30}$$

where

$$\phi = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta - \frac{\tau}{\omega_n}}\right]$$

and

$$\tau = \frac{L_\alpha}{mV}$$

Equation (2-30) shows that the phase and magnitude of the response is determined by ζ and $L_\alpha/mV\omega_n$. These coefficients have been identified

a. import t parameters in the longitudinal handling qualities criteria [6].

Shomber and Gertsen noted the importance of L_α/mv in their study [5]. They argued that when the aircraft's normal acceleration change per unit angle of attack ($N_\alpha = \frac{L_\alpha}{mg}$) is less than 15 g/rad, the pilot is concerned with controlling the aircraft's flight path. The magnitude of the flight path curvature due to elevator deflection is approximately $(L_\alpha/mv)\Delta\alpha$. However when $N_\alpha > 15$ g/rad, the stress experienced by the pilot due to normal acceleration forces causes the pilot to be more concerned with controlling the normal acceleration than controlling the flight path. Therefore Shomber & Gertsen derived two sets of parameters. The first (for $N_\alpha < 15$) is a function of L_α/mv, $\omega_{n_{sp}}$, and ζ_{sp} and the second (for $N_\alpha > 15$) is a function of N_α, $\omega_{n_{sp}}$, and ζ_{sp}.

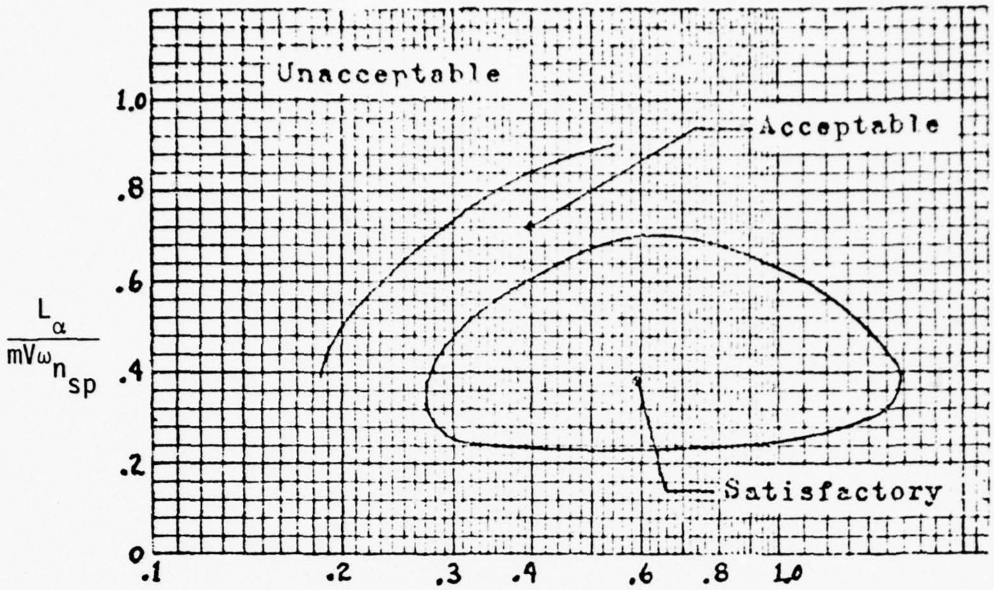
Figures 2-4 and 2-5 show iso-opinion curves based on the use of these parameters. The data used to form these curves is based on pilot ratings from experiments using simulators and variable stability aircraft. The solid lines represent curves of constant pilot rating as the values of $L_\alpha/mv\omega_{n_{sp}}$ or $N_\alpha/\omega_{n_{sp}}$, as the case may be, and ζ_{sp} are varied. The regions of satisfactory, acceptable, and unacceptable handling qualities are indicated.

From Figures 2-4 and 2-5 it is evident that ideally L_α/ω_n, and ζ should be constant for all flight conditions. Since

$$L_\alpha = 1/2\rho V^2 SC_{L_\alpha} = K\rho V^2$$

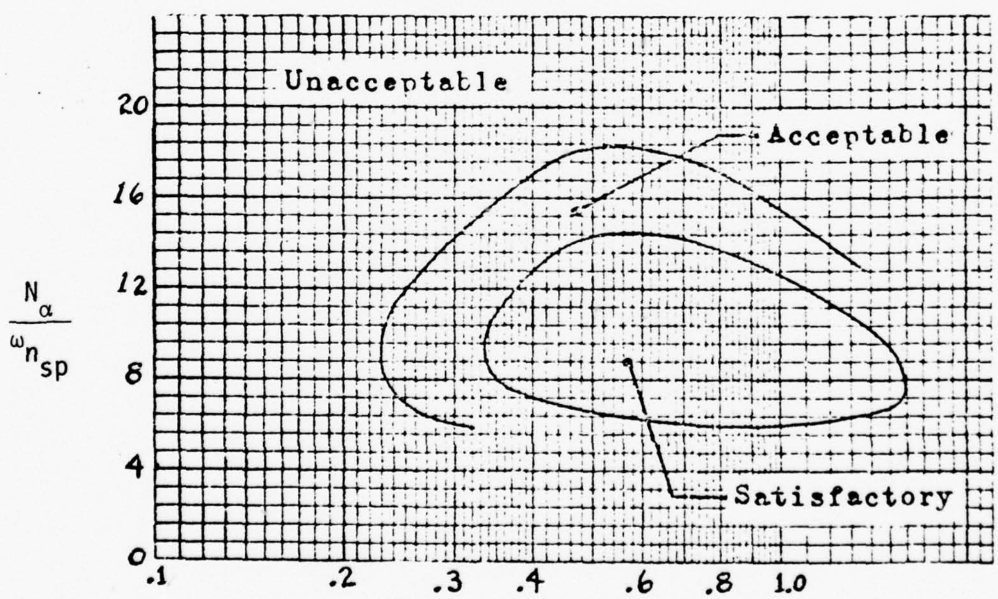
and

$$N_\alpha = \frac{L_\alpha}{mg}$$



ζ_{sp} - Short Period Damping Ratio

Figure 2-4. Performance Criteria Plot When $N_\alpha < 15$.



ζ_{sp} - Short Period Damping Ratio

Figure 2-5. Performance Criteria Plot When $N_\alpha > 15$.

then the compensated system would have to meet the following conditions:

1. ζ is not function of ρ or V.
2. For $N_\alpha < 15$

$$\omega_n = k\rho V \tag{2-31}$$

3. For $N_\alpha > 15$

$$\omega_n = k\rho V^2 \tag{2-32}$$

However, from a practical viewpoint, the previous three rigid conditions do not have to be met as long as the compensated system response remains in the satisfactory region.

If it is assumed that the compensated system will employ state feedback of the form

$$\dot{\underline{x}} = [A + BK]\underline{x} + B\underline{u} \tag{2-33}$$

where the K matrix is a constant feedback matrix, then the characteristic response of the compensated system will be determined by the matrix [A + BK]. Since the elements of both the A and B matrices are dependent on the aerodynamic coefficients for the system, the compensated system response will be a function of the A and B matrices. Recalling that the short period response of the open loop (uncompensated) system is dependent mainly on the $\Delta\alpha$ and q states, it will be assumed that the compensated system's short period response is also dependent only on these states. Using Equation (2-22), the second and fourth rows of the B matrix (Figure 2-3) and the definition for $D\alpha$, $D\hat{q}$, and $\hat{q}$, Equations (2-34) and (2-35) are determined.

$$A = \begin{bmatrix} -\dfrac{2V_e}{\bar{c}}\left[\dfrac{C_{L_\alpha} + C_{D_e}}{2\mu + C_{L_{\dot{\alpha}}}}\right] & \dfrac{2\mu - C_{L_q}}{2\mu + C_{L_{\dot{\alpha}}}} \\ \dfrac{4V_e^2}{\hat{I}_y\bar{c}^2}\left[C_{m_\alpha} - \dfrac{C_{m_{\dot{\alpha}}}(C_{L_\alpha} + C_{D_e})}{2\mu + C_{L_{\dot{\alpha}}}}\right] & \dfrac{2V_e}{\hat{I}_y\bar{c}}\left[C_{m_q} + \dfrac{C_{m_{\dot{\alpha}}}(2\mu - C_{L_q})}{2\mu + C_{L_{\dot{\alpha}}}}\right] \end{bmatrix} \tag{2-34}$$

$$BK = \begin{bmatrix} \dfrac{K_1 V_e}{\bar{c}(2\mu + C_{L_{\dot{\alpha}}})} & \dfrac{K_1 V_e}{\bar{c}(2\mu + C_{L_{\dot{\alpha}}})} \\ \dfrac{K_3 V_e^2}{\hat{I}_y\bar{c}}\left[1 + \dfrac{K_5}{(2\mu + C_{L_{\dot{\alpha}}})}\right] & \dfrac{K_4 V_e^2}{\hat{I}_y\bar{c}}\left[1 + \dfrac{K_5}{2\mu + C_{L_{\dot{\alpha}}}}\right] \end{bmatrix} \tag{2-35}$$

where $\quad K_5 = C_{m_{\dot{\alpha}}} \sin(\alpha_T) + 1$

and K_1, K_2, K_3, and K_4 are constants that are functions of the k_{ij} elements of the feedback matrix. Making the following assumptions,

$$2\mu \gg C_{L_{\dot{\alpha}}}$$

$$2\mu \gg C_{L_q}$$

$$\hat{I}_y = \frac{K_6}{\rho_e}$$

$$\mu = \frac{K_7}{\rho_e}$$

C_{D_e}, $\bar{c}$, C_{m_α}, $C_{m_{\dot{\alpha}}}$, C_{m_q}, C_{L_α} are all constants then Equations (2-34) and (2-35) become

$$A = \begin{bmatrix} K_8\rho_e V_e & 1 \\ K_9\rho_e V_e{}^2 + K_{10}\rho_e{}^2 V_e{}^2 & K_{11}\rho_e V_e \end{bmatrix} \tag{2-36}$$

$$BK = \begin{bmatrix} K_{12}\rho_e V_e & K_{13}\rho_e V_e \\ K_{14}\rho_e V_e{}^2 + K_{15}\rho_e{}^2 V_e{}^2 & K_{16}\rho_e V_e{}^2 + K_{17}\rho_e{}^2 V_e{}^2 \end{bmatrix} \tag{2-37}$$

where K_8 through K_{17} are determined by K_1 through K_7 and the constant aerodynamic coefficients. Adding Equations (2-36) and (2-37),

$$[A+BK] = \begin{bmatrix} K_{18}\rho_e V_e & 1 + K_{13}\rho_e V_e \\ K_{19}\rho_e V_e{}^2 + K_{20}\rho_e{}^2 V_e{}^2 & K_{11}\rho_e V_e + K_{16}\rho_e V_e{}^2 + K_{17}\rho_e{}^2 V_e{}^2 \end{bmatrix} \tag{2-38}$$

where

$$K_{18} = K_8 + K_{12}$$

$$K_{19} = K_9 + K_{14}$$

$$K_{20} = K_{10} + K_{15}$$

Since $\rho_e \ll 1$, then $\rho_e{}^2 \ll \rho_e$ and if it is also assumed that K_{19} and K_{20} and K_{11}, K_{16}, and K_{17} are of approximately the same magnitude, then Equation (2-38) reduces to

$$[A+BK] = \begin{bmatrix} K_{18}\rho_e V_e & 1 + K_{13}\rho_e V_e \\ K_{19}\rho_e V_e{}^2 & K_{16}\rho_e V_e{}^2 \end{bmatrix} \tag{2-39}$$

From Equation (2-39), the manner in which $\omega_{n_{sp}}$ and ζ_{sp} vary with ρ_e and V_e can be determined. Since $2\zeta_{sp}\omega_{n_{sp}}$ is equal to the negative trace of Equation (2-39) and $\omega_{n_{sp}}^2$ is equal to the determinant of the matrix, then

$$2\zeta_{sp}\omega_{n_{sp}} = -[K_{18} + K_{16}]\rho_e V_e = -K_{21}\rho_e V_e \tag{2-40}$$

$$\omega_{n_{sp}}^2 = [K_{16}K_{18} - K_{13}K_{19}]\rho_e^2 V_e^3 - K_{19}\rho_e V_e^2 \tag{2-41}$$

If $[K_{16}K_{18} - K_{13}K_{19}]\rho_e V_e << K_{19}$ then Equation (2-41) reduces to

$$\omega_{n_{sp}} = K_{22}(\rho_e)^{\frac{1}{2}} V_e \tag{2-42}$$

where $K_{22} = -K_{19}$

Substituting Equation (2-42) into Equation (2-40) gives

$$\zeta_{sp} = K_{23}(\rho_e)^{\frac{1}{2}} \tag{2-43}$$

where $K_{23} = -\dfrac{K_{21}}{2K_{22}}$

Although Equations (2-42) and (2-43) do not quite meet the conditions for the ideal system, if the compensator is designed so that at a median velocity and altitude the compensated system's response is well within the satisfactory region, then for a certain limit of flight conditions the system response will remain in the satisfactory region.

Since the location of the zero, L_α/mV, in Equation (2-29) is an important parameter in the performance of the aircraft, it will be useful to determine which elements of the A matrix determine the zero

location. Using the short period approximation matrices, the transfer function can be determined by

$$\frac{q(s)}{\Delta\delta_e(s)} = C[sI-A]^{-1}B \tag{2-44}$$

where

$$C = [0 \quad 1] \tag{2-45}$$

$$A = \begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix} \tag{2-46}$$

$$B = \begin{bmatrix} K_1 b_{22} + K_2 b_{23} \\ K_3 b_{42} + K_4 b_{43} \end{bmatrix} = \begin{bmatrix} b_1' \\ b_2' \end{bmatrix} \tag{2-47}$$

From Equations (2-45), (2-46), and (2-47), Equation (2-44) can be shown to be

$$\frac{q(s)}{\Delta\zeta_e(s)} = \frac{b_2'(s - a_{22}) - b_1' a_{42}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2-48}$$

If $b_2' >> b_1'$, then Equation (2-48) reduces to

$$\frac{q(s)}{\Delta\zeta_e(s)} = \frac{b_2'(s - a_{22})}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2-49}$$

Equation (2-49) shows the zero location is determined by the a_{22} element.

The a_{22} element can be expressed as

$$a_{22} = -\frac{2V}{\bar{c}} \frac{(C_{L_\alpha} + C_{D_e})}{2\mu + C_{L_{\dot{\alpha}}}} \tag{2-50}$$

If it is assumed that

$$2\mu >> C_{L_{\dot{\alpha}}}$$

$$C_{L_\alpha} >> C_{D_e}$$

then Equation (2-50) becomes

$$a_{22} \approx -\frac{VC_{L_\alpha}}{\mu\bar{c}} \tag{2-51}$$

Since,

$$C_{L_\alpha} = \frac{2L_\alpha}{\rho V^2 S}$$

and

$$\mu = \frac{2m}{\rho S\bar{c}}$$

then

$$a_{22} \approx \frac{L_\alpha}{mV} \tag{2-52}$$

Therefore the zero is located at $s = L_\alpha/mv$ which is the same result as given in Equation (2-29).

With the knowledge of how the elements of the A matrix effect the system response, a method for developing a satisfactory model can now be formulated.

1. Through simulation or other means choose a median flight condition for which the compensator will be designed.
2. Pick a desired ζ and ω_n for the phugoid and adjust the a_{13}, a_{21}, and a_{11} elements for the desired response.
3. Determine N_α at the chosen velocity and altitude from the aerodynamic coefficients using $N_\alpha = \frac{L_\alpha}{mg}$.
4. If $N_\alpha < 15$, then determine L_α/mv and calculate an $\omega_{n_{sp}}$ and a ζ_{sp} which place the intersection of $L_\alpha/mv\omega_{n_{sp}}$ and ζ_{sp} in the center of the satisfactory region in Figure 2-4.
5. If $N_\alpha > 15$, then determine an $\omega_{n_{sp}}$ and ζ_{sp} which will place the intersection of $N_\alpha/\omega_{n_{sp}}$ and ζ_{sp} in the center of the satisfactory region in Figure 2-5.
6. Using the desired values of ζ_{sp} and $\omega_{n_{sp}}$, calculate the short period characteristic equation of the model
   $\text{C.E.} = s^2 + 2\zeta_{sp}\omega_{n_{sp}} s + \omega_{n_{sp}}^2$
7. Set $a_{24_m} = 1$
8. Do not change a_{22}. Therefore $a_{22_m} = a_{22_p}$. This is done so the zero location in Equation (2-29) is not changed.
9. Determine a_{44_m} by $a_{44_m} = -(a_{22_m} + 2\zeta_{sp}\omega_{n_{sp}})$
10. Determine a_{42_m} by $a_{42_m} = (a_{22_m} a_{44_m}) - \omega_n^2$.
11. Simulate the model to insure that the model has the desired response.

With the knowledge of how to design a model that will perform as per the performance criteria, the task at hand is to design a compensator which will cause the aircraft to have the same response as the model.

III. DEVELOPMENT OF COMPENSATOR DESIGN ALGORITHMS

Once a model is designed that will perform as per the specified performance criteria, the problem at hand is the design of a compensator that will force the plant to respond in the same manner as the model. There are certain physical restraints with the aircraft problem that must be kept in mind. Consider the block diagram for the mathematical aircraft model as given in Figure 3-1. First, it is assumed that nothing within the dotted outline of Figure 3-1 is accessible. In other words, the A and B matrices cannot be altered directly since they are determined by the configuration of the aircraft and its environment. Second, it is assumed that the states of the aircraft ($\underline{x}$ vector) are available which means that there are devices that can accurately measure the velocity, angle of attack, pitch angle, pitch rate, and altitude. Since the only available information in the system are the inputs and the states, the compensator C(s) must be of the form given in Figure 3-2.

R. T. Curran developed an algorithmic approach for the design of the compensator utilizing his concept of "equicontrollability" [4].

Definition: Given the system

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{3-1}$$

where A has dimension n x n and rank n

B has dimension n x m and rank m

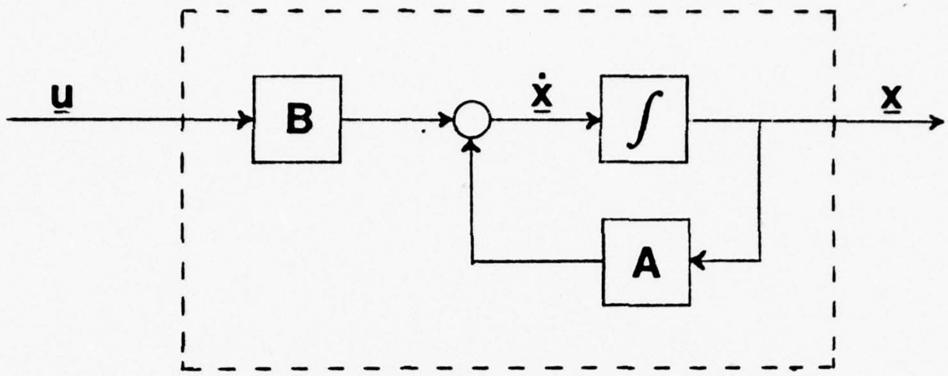


Figure 3-1. System Block Diagram Representation of Mathematical Aircraft Model.

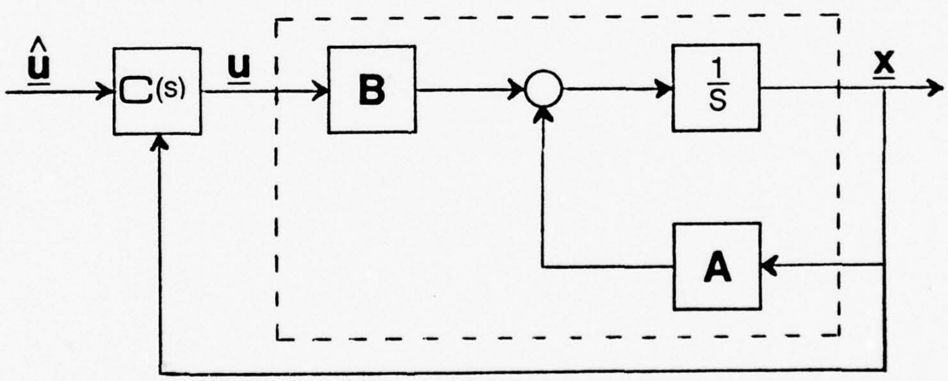


Figure 3-2. System Block Diagram of Proposed Form of Compensation.

the system is controllable if and only if there is a $p \leq n$ such that

$$C_p = [B, AB, \ldots A^{p-1}B] \tag{3-2}$$

has full rank n = dimension of A and furthermore the pair (A,B) has controllability index p if p is the smallest integer that satisfies Equation (3-2).

Definition: The pair (A,B) is equicontrollable if n = pm where p is the controllability index of (A,B), A is n x n, and B is n x m.

Note, the definition of equicontrollability implies that the first n = pm columns of C_p are independent.

Based on Luenberger's work on canonical forms for multivariable systems [7], Curran showed that if a system is equicontrollable then there exists a similarity transformation T such that

$$T^{-1}AT = \begin{array}{cc} \begin{array}{cc} m & n-m \end{array} & \\ \left[\begin{array}{c|c} \theta & I \\ \hline \multicolumn{2}{c}{X} \end{array}\right] & \begin{array}{c} n-m \\ m \end{array} \end{array} \tag{3-3}$$

and

$$T^{-1}B = \begin{array}{cc} \left[\begin{array}{c} \theta \\ \hline I \end{array}\right] & \begin{array}{c} n-m \\ m \end{array} \end{array} \tag{3-4}$$

The restriction, that for a system to be equicontrollable, (n=pm) seems to be severe and therefore of little practical usefulness. Indeed,

most systems will not meet this structural constraint. To overcome this problem, Curran proposes to add states to the system that will alter the structure of the system but not it's transfer function. Such states will have to be controllable but not observable. Curran proves the following:

Given the system (A,B) with controllability index p, A is n x n, B is n x m. Then if n < pm, there exists matrices J and L and a number N such that the N-dimensional system

$$\left(\left[\begin{array}{c|c} A & \theta \\ \hline J & L \end{array} \right] , \left[\begin{array}{c} B \\ \hline \theta \end{array} \right] \right) \triangleq (\hat{A},\hat{B})$$

is equicontrollable if and only if $N = m(p + k)$ for some integer $k \geq 0$.

The manner in which the additional k states are connected to the original system will determine the J matrix and the pole-placement of the additional states will determine the L matrix.

Before pursuing the determination of the J and L matrices it will prove useful to examine the structure of an equicontrollable system in canonical form. Consider the following example.

<u>Example 3.1</u>

Given the equicontrollable canonical system (A,B) with controllability index p, A is n x n, B is n x m

$$A = \left[\begin{array}{c|c} \theta & I \\ \hline \multicolumn{2}{c}{X} \end{array}\right]\begin{array}{l} (p-1)m \\ m \end{array}.$$

$$B = \left[\begin{array}{c} \theta \\ \hline I \end{array}\right]\begin{array}{l} (p-1)m \\ m \end{array}$$

let n = 6, m = 3 then p = 2 and

$$A = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{array}\right]$$

$$B = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

the state equations for the system can be written

$$\dot{x}_1 = x_4$$
$$\dot{x}_2 = x_5$$
$$\dot{x}_3 = x_6$$

$$\dot{x}_4 = f_1(\underline{x},\underline{u})$$
$$\dot{x}_5 = f_2(\underline{x},\underline{u})$$
$$\dot{x}_6 = f_3(\underline{x},\underline{u}) \quad .$$

From the state equation a system block diagram can be readily drawn as shown in Figure 3-3.

There are 3 strings of integrators, each consisting of two integrators. In general, for an equicontrollable system in canonical form there will be m strings of integrators of length p. In other words, each of the inputs controls an equal number of states.

Curran shows that given the system matrices (A,B), where A is n x n and B is n x m and full rank, then there exists a similarity transformation T such that

$$T^{-1}AT = \begin{array}{cc} \begin{array}{cc} m & n-m \end{array} & \\ \left[\begin{array}{c|c} \theta & I \\ \hline \multicolumn{2}{c}{X} \end{array}\right] & \begin{array}{c} n-m \\ m \end{array} \end{array} \tag{3-5}$$

$$T^{-1}B = \begin{array}{cc} \left[\begin{array}{c} \theta \\ \hline W \end{array}\right] & \begin{array}{c} n-m \\ m \end{array} \end{array} \tag{3-6}$$

if and only if the last n columns of the controllability matrix

$$C = [A^{n-1}B,\ A^{n-2}B,\ldots AB,B]$$

are linearly independent.

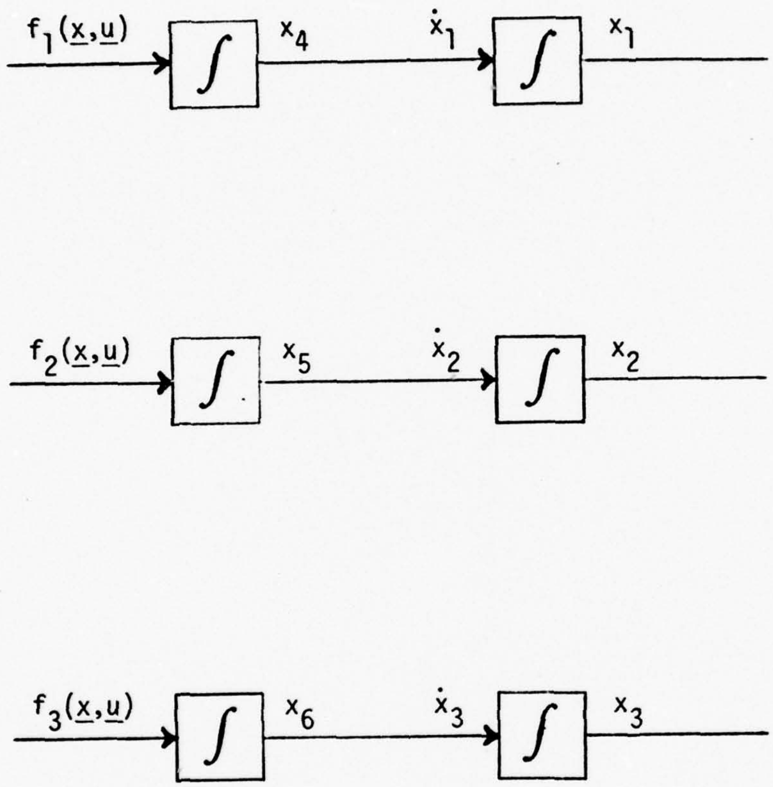


Figure 3-3. Block Diagram Representation of State Equations for the Equicontrollable System Given in Example 3.1.

Once the system has been transformed into the form of equations (3-5) and (3-6) the states can easily be added so the system is in equicontrollable form. Example 3.2 illustrates this process.

Example 3.2

Given the system matrices (F,G) in the form of Equations (3-5) and (3-6) where F is 5x5, G is 5x3

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ & & \text{---} f_1 \text{---} & & \\ & & \text{---} f_2 \text{---} & & \\ & & \text{---} f_3 \text{---} & & \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ & \text{---} g_1 \text{---} & \\ & \text{---} g_2 \text{---} & \\ & \text{---} g_3 \text{---} & \end{bmatrix}$$

then

$$\dot{x}_1 = x_4$$

$$\dot{x}_2 = x_5$$

$$\dot{x}_3 = f_1(\underline{x}) + g_1(\underline{u})$$

$$\dot{x}_4 = f_2(\underline{x}) + g_2(\underline{u})$$

$$\dot{x}_5 = f_3(\underline{x}) + g_3(\underline{u})$$

and the system block diagram will be the form as shown in Figure 3-4. From the block diagram it is obvious that the addition of one integrator will make the system equicontrollable. The state (x_6) should be added such that

$$\dot{x}_6 = x_3 + \alpha x_6 \tag{3-7}$$

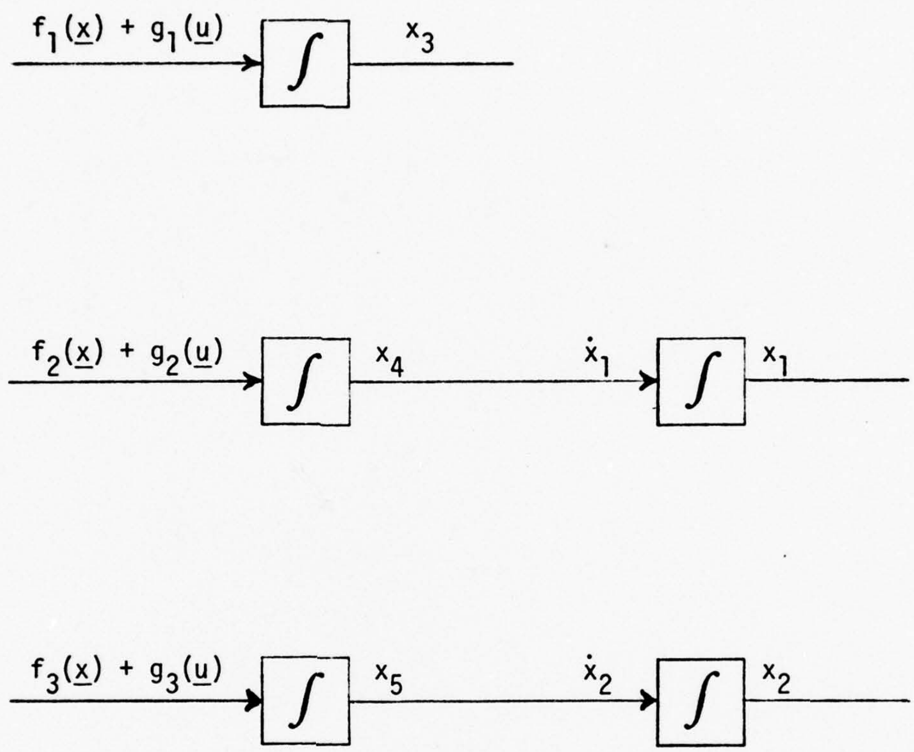


Figure 3-4. Block Diagram Representation of State Equations for the System Given in Example 3.2.

where α is the pole location of the added state. The augmented system matrices $(\hat{F},\hat{G})$ are now in the form

$$\hat{F} = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & 0 \\ & & F & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & \alpha \end{array}\right] \quad \text{and} \quad \hat{G} = \left[\begin{array}{ccc} & & \\ & & \\ & G & \\ & & \\ & & \\ \hline 0 & 0 & 0 \end{array}\right] \tag{3-8}$$

Hence,

$$J = [0 \quad 0 \quad 1 \quad 0 \quad 0] \tag{3-9}$$

and

$$L = [\alpha] \tag{3-10}$$

Once the system is expanded such that it is equicontrollable then the system can once again be transformed so the final system matrices are in equicontrollable canonical form.

So far it has only been shown that there exists a similarity transformation matrix T that will transform the system matrices into the desired form. The process for generating this transformation matrix T is straightforward and is outlined in the following steps.

1. Given the system matrices (A,B) with controllability index p, where A is nxn and B is nxm, generate the matrix P from the last n columns* of the partial controllability matrix C where

---

*(the last n columns must be taken so the transformed system will be of the form given in Equation (3-5).)

$$C = [A^{p-1}B,\ A^{p-2}B, \ldots, AB, B] \tag{3-11}$$

2. Compute the matrix Q such that

$$Q = P^{-1} \tag{3-12}$$

3. Take the first m rows of Q to form the mxn matrix E and partition it such that

$$E = \begin{array}{c} n \\ \left[\begin{array}{c} E_1 \\ \hline E_2 \end{array}\right] \end{array} \begin{array}{l} \\ r \\ m-r \end{array} \tag{3-13}$$

where $r = n - (p-1)m$

4. Form the matrix S such that

$$S = \begin{bmatrix} E \\ EA \\ \vdots \\ EA^{p-2} \\ E_1A^{p-1} \end{bmatrix} \tag{3-14}$$

5. Determine the transformation matrix T where

$$T = S^{-1} \tag{3-15}$$

Note that the above algorithm works for all systems that meet the requirement that the last n columns of the controllability matrix are linearly

independent. For an equicontrollable system $n = pm$ and $m-r$ in Equation (3-13) would equal zero.

The purpose of augmenting the system to achieve equicontrollability and then transforming it to a canonical form is that the structure of the transformed system can be utilized to easily find a state feedback matrix K that will alter the pole and zero locations of the original system. The problem is outlined below.

Given the plant and model state equations

$$\dot{\underline{x}}_p = A_p\underline{x}_p + B_p\underline{u} \tag{3-16}$$

$$\dot{\underline{x}}_m = A_m\underline{x}_m + B_m\underline{u} \tag{3-17}$$

where

dimension of A_p and A_m = nxn

dimension of B_p and B_m = nxm

determine the feedback matrix K such that the transfer function of the compensated plant equals the transfer function of the model as given by Equation (3-18).

$$[sI - (A_p + B_pK)]^{-1}B_p = [sI - A_m]^{-1}B_m \tag{3-18}$$

The process for determining the matrix K involves parallel operations on the plant and model state equations beginning with the original systems given by Equations (3-16) and (3-17). An overview of the algorithm is given on the following page.

1. Transform the plant and model to a canonical form as given by Equations (3-5) and (3-6).
2. Add states to the plant and model as outlined in Example 3-2.
3. Transform the augmented plant and model systems to equicontrollable form.
4. If the plant and model similarity transformations used in 1 and 3 are equal, then solve directly for the feedback matrix K.
5. Transform the compensated plant back to the original coordinates.

Note from 4, the feedback matrix K can be solved directly if the plant and model similarity transformations are equal. It will be shown later that for aircraft longitudinal equations the similarity transformations will always be the same if care is taken in constructing the model. For systems that do not meet this requirement, Curran's algorithm completes the procedure for developing the compensator.

The Algorithm in Detail

Given the state equations for the plant

$$\dot{\underline{x}}_p = A_p \underline{x}_p + B_p \underline{u} \tag{3-19}$$

transform the system to a canonical form so the additional states can be added. Let

$$\underline{x}_p = T_{p_1} \hat{\underline{x}}_p \tag{3-20}$$

Substituting Equation (3-20) into Equation (3-19) gives

$$T_{p_1}\dot{\hat{\underline{x}}}_p = A_p T_{p_1}\hat{\underline{x}}_p + B_p\underline{u} \tag{3-21}$$

$$\dot{\hat{\underline{x}}}_p = T_{p_1}^{-1} A_p T_{p_1}\hat{\underline{x}}_p + T_{p_1}^{-1} B_p\underline{u} \tag{3-22}$$

$$\dot{\hat{\underline{x}}}_p = \hat{A}_p\hat{\underline{x}}_p + \hat{B}_p\underline{u} \tag{3-23}$$

where

$$\hat{A}_p = T_{p_1}^{-1} A_p T_{p_1} = \begin{array}{cc} \begin{array}{cc} m & (n-m) \end{array} & \\ \left[\begin{array}{c|c} \theta & I \\ \hline \multicolumn{2}{c}{X} \end{array}\right] & \begin{array}{c} (n-m) \\ m \end{array} \end{array} \tag{3-24}$$

and

$$\hat{B}_p = T_{p_1}^{-1} B_p = \begin{array}{cc} \left[\begin{array}{c} \theta \\ \hline W \end{array}\right] & \begin{array}{c} n-m \\ m \end{array} \end{array} \tag{3-25}$$

Augment the above equations such that the new system is equicontrollable (N=pm) as shown in Figure 3-5 and given by Equation (3-26). Let

$$\overline{\underline{x}}_p = \begin{array}{cc} \left[\begin{array}{c} \hat{\underline{x}}_p \\ \hline \underline{z}_p \end{array}\right] & \begin{array}{c} n \\ r \end{array} \end{array} \tag{3-26}$$

then the state equations become

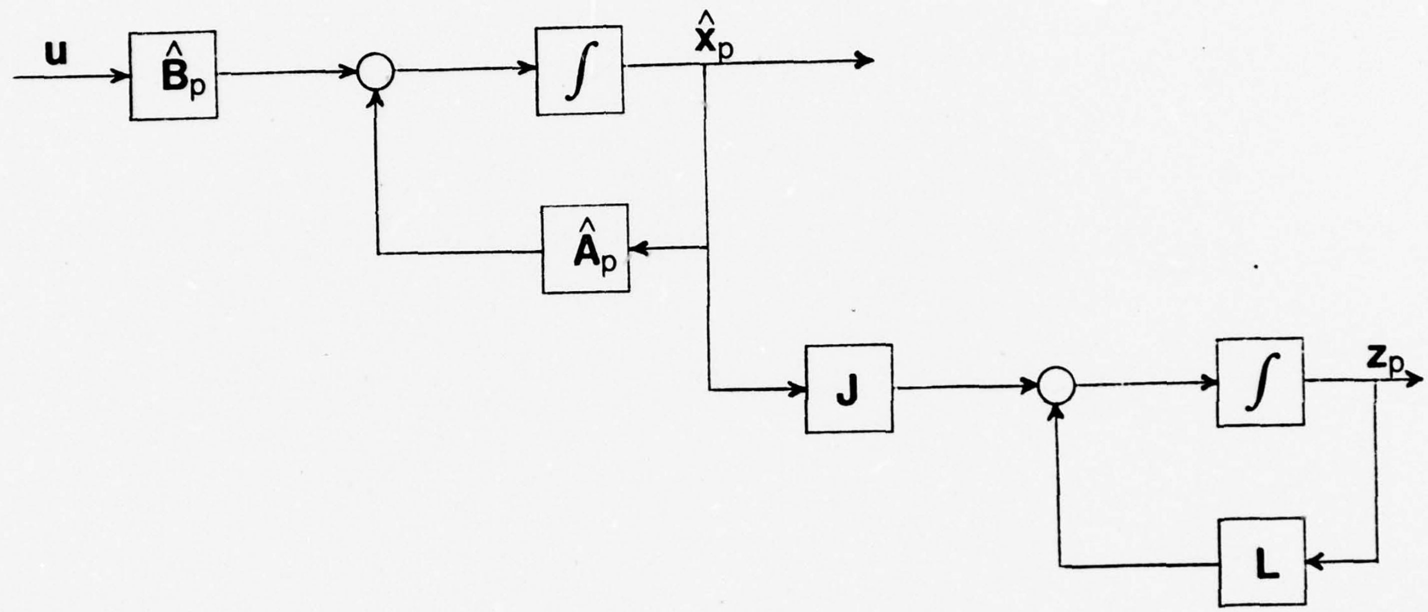


Figure 3-5. Block Diagram Showing the Method of Connecting the Added States to the Original System.

$$\dot{\underline{\bar{x}}} = \begin{array}{c} \begin{array}{cc} n & r \end{array} \\ \left[\begin{array}{c|c} \hat{A} & \theta \\ \hline J & L \end{array}\right] \end{array} \underline{\bar{x}} + \left[\begin{array}{c} \hat{B} \\ \hline \theta \end{array}\right] \underline{u} \tag{3-27}$$

$$\dot{\underline{\bar{x}}}_p = \bar{F}_p \underline{\bar{X}}_p + \bar{G}_p \underline{u} \tag{3-28}$$

where

$$(n + r) = N = pm$$

The J and L matrices are determined by the method given in Example 3.2.

Next transform the augmented system to the equicontrollable canonical form via similarity transformation T_{p_2}. Now the system equations are

$$\dot{\underline{\bar{x}}}_p{}^* = \bar{F}_p{}^* \underline{\bar{x}}_p{}^* + \bar{G}_p{}^* \underline{u} \tag{3-29}$$

where

$$\underline{\bar{x}}_p = \bar{T}_{p_2} \underline{\bar{x}}_p{}^* \tag{3-30}$$

$$\bar{F}_p{}^* = T_{p_2}{}^{-1} \bar{F}_p T_{p_2} \tag{3-31}$$

$$\bar{G}_p^* = T_{p_2}{}^{-1} \bar{G}_p \tag{3-32}$$

$$\bar{F}_p{}^* = \begin{array}{cc} \begin{array}{cc} m & (p-1)m \end{array} & \\ \left[\begin{array}{c|c} \theta & I \\ \hline \multicolumn{2}{c}{X_p} \end{array}\right] & \begin{array}{l} (p-1)m \\ m \end{array} \end{array} \tag{3-33}$$

$$\overline{G}_p^* = \begin{array}{c} \left[\begin{array}{c} \theta \\ \hline I \end{array}\right] \end{array} \begin{array}{l} (p-1)m \\ m \end{array} \tag{3-34}$$

Once the plant has been augmented and transformed to equicontrollable canonical form, repeat the same process with the model. Once the system model has been augmented and transformed it will be of the form

$$\dot{\underline{\overline{x}}}_m^* = \overline{F}_m^* \underline{\overline{x}}_m^* + \overline{G}_m^* \underline{u} \tag{3-35}$$

$$\overline{F}_m^* = \begin{array}{c} \begin{array}{cc} m & (p-1)m \end{array} \\ \left[\begin{array}{c|c} \theta & I \\ \hline \multicolumn{2}{c}{X_m} \end{array}\right] \end{array} \begin{array}{l} \\ (p-1)m \\ m \end{array} \tag{3-36}$$

$$\overline{G}_m^* = \left[\begin{array}{c} \theta \\ \hline I \end{array}\right] \begin{array}{l} (p-1)m \\ m \end{array} \tag{3-37}$$

If $T_{p_1} = T_{m_1}$ and $T_{p_2} = T_{m_2}$, then the K matrix can be determined directly as shown in the following paragraph.

Given the equicontrollable canonical system

$$\dot{\underline{\overline{x}}}_p^* = \overline{F}_p^* \underline{\overline{x}}_p^* + \overline{G}_p^* \underline{u} \tag{3-38}$$

let

$$\underline{u} = \hat{\underline{u}} + \overline{K}^* \underline{\overline{x}}_p^* \tag{3-39}$$

Substituting Equation (3-39) into Equation (3-38) gives

$$\dot{\underline{\overline{x}}}_p^* = (\overline{F}_p^* + \overline{G}_p^* \overline{K}^*) \underline{\overline{x}}_p^* + \overline{G}_p^* \hat{\underline{u}} \tag{3-40}$$

The goal up to now has been to make the transfer function of the compensated plant equal to the transfer function of model such that

$$[sI - (A_p + B_pK)]^{-1} B_p = [sI - A_m]^{-1} B_m \tag{3-41}$$

Utilizing the fact that the similarity transformations used for the plant and model are the same and $\overline{G}_p^* = \overline{G}_m^*$ (it will be shown that this implies $B_m = B_p$), the problem is now to find a $\overline{K}^*$ matrix such that

$$[sI - (\overline{F}_p^* + \overline{G}_p^*\overline{K}^*)]^{-1} = [sI - \overline{F}_m^*]^{-1} \tag{3-42}$$

resulting in

$$\overline{F}_p^* + \overline{G}_p^*\overline{K}^* = \overline{F}_m^* \tag{3-43}$$

Because of the structure of $\overline{G}_p^*$

$$\overline{K}^* = X_m - X_p \tag{3-44}$$

where X_p and X_m are the mxpm submatrices given in Equations (3-33) and (3-36) respectively. Once $\overline{K}^*$ is determined then the system is returned to the original coordinates. The process for returning the original coordinates is tedious but straightforward. Since the plant and model transformation matrices were equal, for convenience the subscripts will be dropped. In order to separate the added states from the states of the original system, T_2^{-1} is partitioned such that

$$T_2^{-1} = \begin{array}{c} \begin{array}{cc} \quad n & \quad r \end{array} \\ \left[\begin{array}{c|c} T_{21}^{-1} & T_{22}^{-1} \end{array}\right] \end{array} \tag{3-45}$$

and

$$\underline{\overline{x}}^* = T_{21}^{-1}\hat{\underline{x}} + T_{22}^{-1}\underline{z} \tag{3-46}$$

The state equations for the compensator in the original coordinates are

$$\dot{\underline{z}} = L\underline{z} + JT_1^{-1}\underline{x} \tag{3-47}$$

$$\underline{u} = \hat{\underline{u}} + \overline{K}^* T_{21}^{-1} T_1^{-1}\underline{x} + \overline{K}^* T_{22}^{-1}\underline{z} \tag{3-48}$$

The block diagram for the compensator is given in Figure 3-6 and the compensated system is shown in Figure 3-2. The following example demonstrates the algorithm in detail.

Example 3.3

Given the plant and model matrices.

$$A_p = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 1 \\ -1 & 2 & -1 \end{bmatrix} \qquad B_p = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$A_m = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -2 & 1 \\ 1 & -1 & -2 \end{bmatrix} \qquad B_m = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

The controllability index $p = 2$. This means that $N = pm = 4$; therefore one state will have to be added so that the system is equi-controllable.

First the system will be transformed to a canonical form using a similarity transformation matrix T_{p_1} in order to determine how to

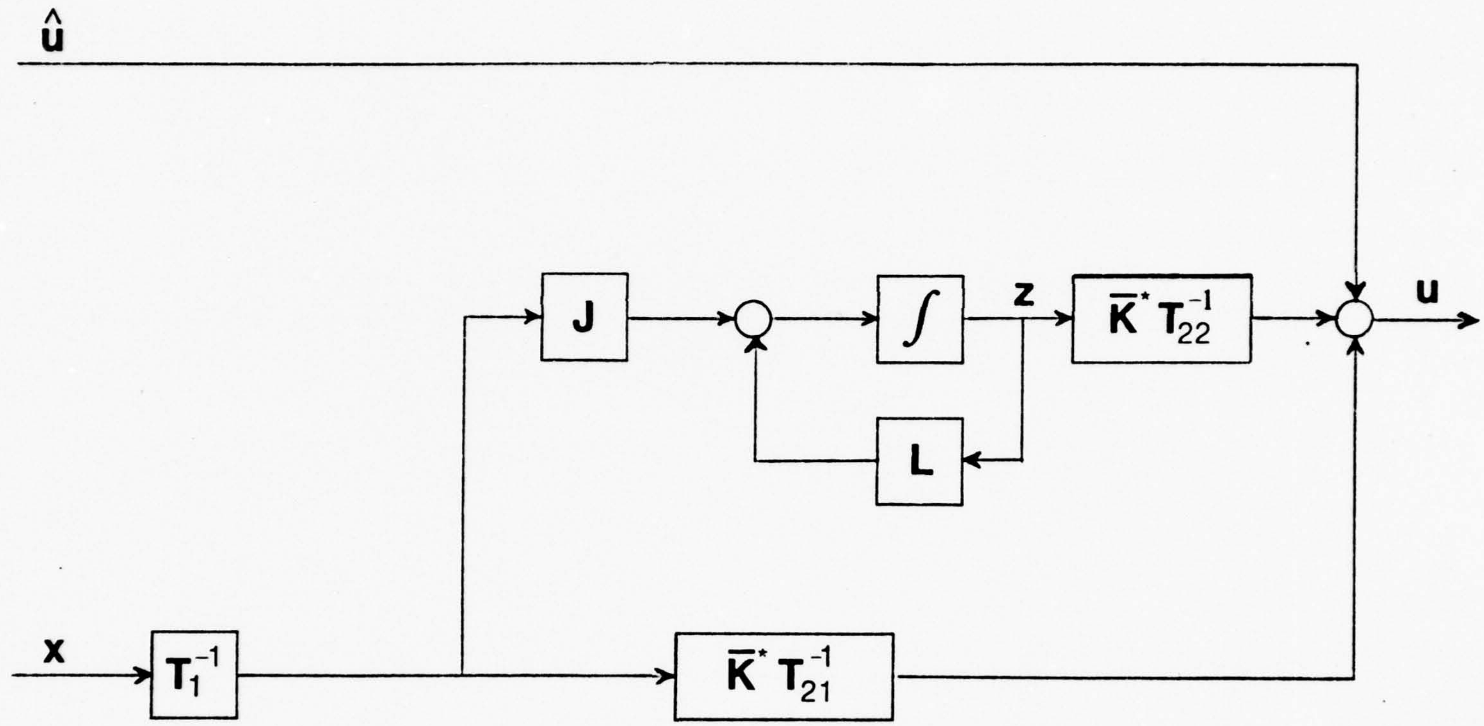


Figure 3-6. Block Diagram Representation of the Compensator.

connect the additional state. Transformation matrix T_{p_1} is generated using Equations (3-11) through (3-15). From Equation (3-11)

$$C_p = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 3 & 5 & 1 & 1 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

The matrix P is taken as the last n rows of C_p.

$$P = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

From Equation (3-12)

$$Q = P^{-1} = \begin{bmatrix} .5 & 0 & 0 \\ -2.5 & 1 & -.5 \\ 0 & 0 & .5 \end{bmatrix}$$

From Equation (3-13)

$$E = \begin{bmatrix} .5 & 0 & 0 \\ -2.5 & 1 & -.5 \end{bmatrix}$$

and

$$E_1 = \begin{bmatrix} .5 & 0 & 0 \end{bmatrix}$$

From Equation (3-14)

$$S = \begin{bmatrix} .5 & 0 & 0 \\ -2.5 & 1 & -.5 \\ -.5 & 1 & 0 \end{bmatrix}.$$

From Equation (3-15)

$$T_{p_1} = S^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ -8 & -2 & 2 \end{bmatrix}$$

Using T_{p_1} the original plant is transformed to the new coordinate system $\hat{\underline{x}}_p$ where

$$\underline{x}_p = T_{p_1}\hat{\underline{x}}_p$$

Using Equation (3-24)

$$\hat{A}_p = T_{p_1}{}^{-1}A_pT_{p_1} = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -3 & 0 \\ -3 & -2 & 4 \end{bmatrix} \tag{3-49}$$

Using Equation (3-25)

$$\hat{B}_p = T_{p_1}{}^{-1}B_p = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{3-50}$$

Using the process as demonstrated in Example 3.2 and the state equations from Equations (3-49) and (3-50) the block diagram for the system can

be drawn as shown in Figure 3-7. From the block diagram it is obvious that the added state z_1 will have to satisfy

$$\dot{z}_1 = \hat{x}_2 \tag{3-51}$$

The pole of this added state is placed at s = -10. Therefore

$$J = [0 \quad 1 \quad 0] \tag{3-52}$$

and

$$L = [-10] \tag{3-53}$$

The augmented system is now

$$\dot{\underline{\bar{x}}}_p = \bar{F}_p \underline{\bar{x}}_p + \bar{G}_p \underline{u} \tag{3-54}$$

$$\dot{\underline{\bar{x}}}_p = \left[\begin{array}{c} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \hline \dot{z}_1 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 \\ -3 & -2 & 4 & 0 \\ \hline 0 & 1 & 0 & -10 \end{array}\right] \left[\begin{array}{c} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hline z_1 \end{array}\right] + \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ \hline 0 & 0 \end{array}\right] \underline{u} \tag{3-55}$$

Next, the equicontrollable system is transformed to the equicontrollable canonical form via similarity transformation matrix T_{p_2}. Matrix T_{p_2} is generated by the same algorithm as matrix T_{p_1}.

$$T_{p_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 10 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{3-56}$$

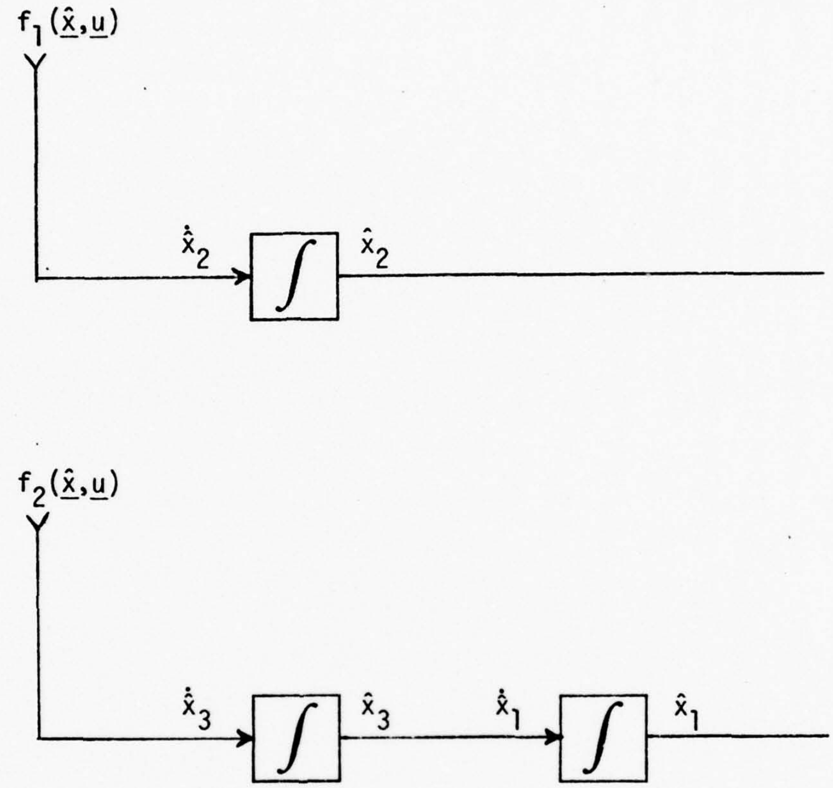


Figure 3-7. Block Diagram Representation of State Equations for the System in Canonical Form in Example 3.3.

$$\bar{F}_p^* = \bar{T}_{p_2}^{-1}\bar{F}_p\bar{T}_{p_2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -35 & -5 & -13 & 0 \\ 12 & 2 & 15 & 4 \end{bmatrix} \quad (3\text{-}57)$$

$$\bar{G}_p^* = T_{p_2}^{-1}\bar{G}_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3\text{-}58)$$

The same process is now performed on the model. Since the solution is dependent on the necessity that $T_{p_1} = T_{m_1}$, let $T_{m_1} = T_{p_1}$ and perform the transformation.

$$T_{m_1} = T_{p_1} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ -8 & -2 & 2 \end{bmatrix} \quad (3\text{-}59)$$

If the transformed model is of the desired form, the algorithm is satisfied. If the transformed model is not of the desired form, then there is no solution and the algorithm should be terminated. The model is then transformed using $T_{m_1}=T_{p_1}$

$$\hat{A}_m = T_{m_1}^{-1}A_mT_{m_1} = \begin{bmatrix} 0 & 0 & 1 \\ -20.5 & -4 & -2.5 \\ -12 & -2 & -1 \end{bmatrix} \; ; \; \hat{B}_m = T_{m_1}^{-1}B_m = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad (3\text{-}60)$$

The model is now augmented so the system is equicontrollable. It is important that the pole location of the added state for the model is the same as that of the plant to insure that the similarity transformation matrix T_{m_2} equals T_{p_2}. The augmented system is now

$$\dot{\bar{\underline{x}}}_m = \bar{F}_m \bar{\underline{x}}_m + \bar{G}_m \underline{u} \tag{3-61}$$

$$\dot{\bar{\underline{x}}}_m = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -20.5 & -4 & -2.5 & 0 \\ -12 & -2 & -1 & 0 \\ 0 & 1 & 0 & -10 \end{bmatrix} \bar{\underline{x}}_m + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \underline{u} \tag{3-62}$$

T_{m_2} is determined and the augmented model is transformed to equicontrollable canonical form.

$$T_{m_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 10 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{3-63}$$

$$\bar{F}_m^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -60.5 & -20.5 & -16.5 & -2.5 \\ 28.5 & 8.5 & 13.5 & 1.5 \end{bmatrix} \tag{3-64}$$

$$\overline{B}_m^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3-65}$$

Since $T_{p_1} = T_{m_1}$ and $T_{p_2} = T_{m_2}$ the feedback matrix $\overline{K}^*$ can be determined directly. From Equation (3-57)

$$X_p = \begin{bmatrix} -35 & -5 & -13 & 0 \\ 12 & 2 & 15 & 4 \end{bmatrix} \tag{3-66}$$

From Equation (3-64)

$$X_m = \begin{bmatrix} -60.5 & -20.5 & -16.5 & -2.5 \\ 28.5 & 8.5 & 13.5 & 1.5 \end{bmatrix} \tag{3-67}$$

Using Equation (3-44)

$$\overline{K}^* = X_m - X_p = \begin{bmatrix} -25.5 & -15.5 & -3.5 & -2.5 \\ 16.5 & 6.5 & -1.5 & -2.5 \end{bmatrix} \tag{3-68}$$

To transform the compensator back to the original coordinates the matrices L, JT_1^{-1}, $\overline{K}^*T_{21}^{-1}T_1^{-1}$, $\overline{K}^*T_{22}^{-1}$ must be determined. First T_2^{-1} is partitioned as per Equation (3-45) to determine T_{21}^{-1} and T_{22}^{-1}.

$$T_2^{-1} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -10 \\ 0 & -1 & 1 & 10 \end{array}\right] \tag{3-69}$$

From Equation (3-69)

$$T_{21}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{3-70}$$

$$T_{22}^{-1} = \begin{bmatrix} 1 \\ -1 \\ -10 \\ 10 \end{bmatrix} \tag{3-71}$$

Performing the required matrix operations the results are:

$$L = [-10] \tag{3-72}$$

$$JT_1^{-1} = [-2.5 \quad 1 \quad -.5] \tag{3-73}$$

$$\overline{K}^* T_{21}^{-1} T_1 = \begin{bmatrix} -4 & -3.5 & .5 \\ 1 & -1.5 & -.5 \end{bmatrix} \tag{3-74}$$

$$\overline{K}^* T_{22}^{-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3-75}$$

The compensator in matrix form is

$$\dot{z}_1 = [-10] z_1 + [-2.5 \quad 1 \quad -.5] \underline{x}_p \tag{3-76}$$

$$\underline{u} = \left[\begin{array}{cc|ccc} 1 & 0 & -4 & -3.5 & .5 \\ 0 & 1 & 1 & -1.5 & -.5 \end{array}\right] \begin{bmatrix} \hat{\underline{u}} \\ \hline \underline{x}_p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} z_1 \tag{3-77}$$

Note from Equation (3-77) that the output of the compensator $\underline{u}$ is not a function of the additional state z_1, but only of the input and the system states. Therefore there is no need to generate z_1 and the system can be compensated by pure state feedback. The final form of the compensated system is shown in Figure 3-8 and the state equations for the compensated system are

$$\dot{\underline{x}}_p = [A_p + B_pK]\underline{x}_p + B_p\hat{\underline{u}} \tag{3-78}$$

where

$$A_p + B_pK = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -2 & 1 \\ 1 & -1 & -2 \end{bmatrix} = A_m$$

Since

$$A_p + B_pK = A_m$$

and

$$B_p = B_m$$

then

$$[sI - (A_p + B_pK)]^{-1}B_p = [sI - A_m]^{-1}B_m \tag{3-79}$$

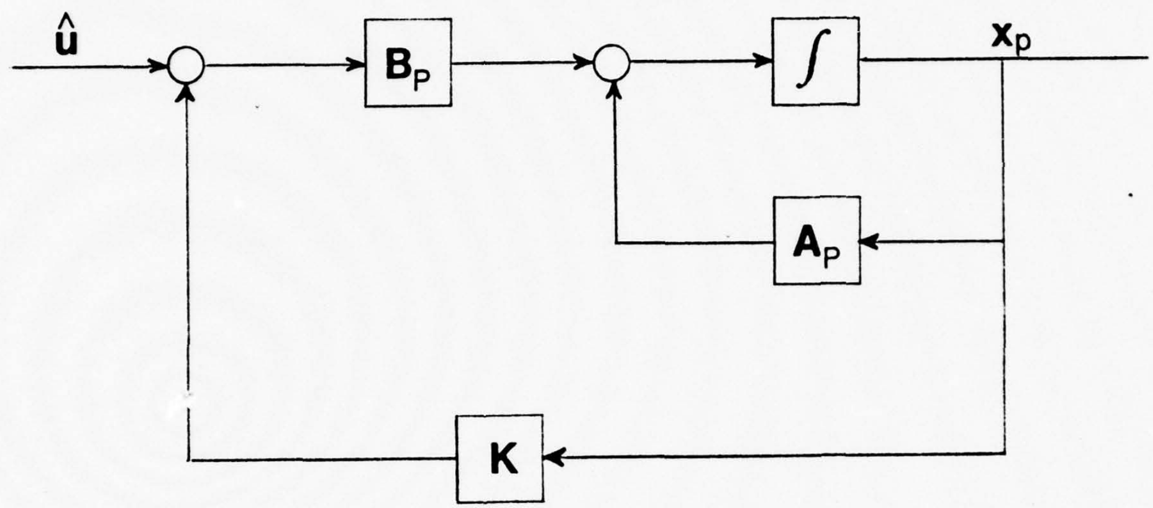


Figure 3-8. Final Form of Compensated System in Example 3.3.

Example 3.3 was a contrived problem, however it illustrates a fundamental flaw in Curran's algorithm. If T_{m_1} had been generated using the method given by Equations (3-11) through (3-15), T_{m_1} would not have equaled T_{p_1} which would have created considerably more work in solving the problem. An alternate method which will preclude the possibility of this happening is proposed below.

Proposition 1

Consider the system pair (A,B) such that

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

where A has dimension n x n and B has dimension n x m and both (A,B) are full rank and the system is controllable with controllability index p. If $n \neq pm$ then add r states to the system such that $N = n + r = pm$ and the augmented system is controllable with controllability index p.

Proposition 1 removes the need for transformation matrix T_1. However, there is still the possibility that T_{p_2} will not equal T_{m_2}. The designer of course could simply let $T_{m_2} = T_{p_2}$ as was done in Example 3.3 with T_{m_1}. However, it would be nice if there were some criteria for insuring that $T_{m_2} = T_{p_2}$. An examination of the similarity transformation and the expected results will be useful for this purpose.

Curran showed that given the equicontrollable system

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

with controllability index p then the transformation matrix T must transform the system such that

$$T^{-1}AT = \begin{array}{c} \begin{array}{cc} m & n-m \end{array} \\ \left[\begin{array}{c|c} \theta & I \\ \hline \multicolumn{2}{c}{X} \end{array}\right] \end{array} \begin{array}{l} \\ n-m \\ m \end{array} \tag{3-80}$$

and

$$T^{-1}B = B^{*} = \left[\begin{array}{c} \theta \\ \hline I \end{array}\right] \begin{array}{l} n-m \\ m \end{array} \tag{3-81}$$

From Equation (3-81)

$$TT^{-1}B = TB^{*} \tag{3-82}$$

$$\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ t_1 & t_2 & \cdots & t_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{3-83}$$

therefore from (3-83), the last m columns of T must equal the columns of B.

$$T = \begin{bmatrix} | & | & & | & | & | & & | \\ t_1 & t_2 & \cdots & t_{n-m} & b_1 & b_2 & \cdots & b_m \\ | & | & & | & | & | & & | \end{bmatrix} \tag{3-84}$$

Using the results from Equation (3-84) and Equation (3-80)

$$TT^{-1}AT = TA^{*} \tag{3-85}$$

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} | & & | & | & & | \\ t_1 & \cdots & t_{n-m} & b_1 & \cdots & b_m \\ | & & | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & & | \\ t_1 & \cdots & t_{n-m} & b_1 & \cdots & b_m \\ | & & | & | & & | \end{bmatrix} \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & 1 & 0 & 0 \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \cdots & 1 \\ \hline & & & X & & \end{array}\right] \tag{3-86}$$

Equation (3-84) shows that for the plant and model transformations to be equal, the input matrices of the two systems must be equal. Hence a necessary but not sufficient criterion for $T_{p_2} = T_{m_2}$ is

$$B_m = B_p \tag{3-87}$$

Equation (3-86) shows that the remaining terms of the similarity transformation depend on the elements of the A and B matrices.

If the system is of a particular form, Equation (3-86) will be useful in showing which elements of the A matrix determine the transformation matrix.

<u>THEOREM I</u>

Given the equicontrollable system (A,B) with controllability index p = 2 where A is dimension n x n and B is dimension n x m and full rank.

If B has n - m zero rows then the similarity transformation matrix T that transforms the system to equicontrollable canonical form

will depend only on the B matrix and the row elements of the homogeneous states in the A matrix.

<u>Proof</u>

Given

$$A = \begin{array}{c} \begin{array}{cc} m & m \end{array} \\ \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \end{array} \begin{array}{l} \\ m \\ m \end{array} \quad \text{and } B = \begin{array}{c} m \\ \left[\begin{array}{c} \theta \\ \hline B_1 \end{array}\right] \end{array} \begin{array}{l} \\ m \\ m \end{array} \tag{3-88}$$

then as shown in Equation (3-84)

$$T = \begin{array}{c} \begin{array}{cc} m & m \end{array} \\ \left[\begin{array}{c|c} T_1 & \theta \\ \hline T_2 & B_1 \end{array}\right] \end{array} \begin{array}{l} \\ m \\ m \end{array} \tag{3-89}$$

Since $T^{-1}AT = A^*$ where A^* is the A matrix in equicontrollable canonical form, then $AT = TA^*$ as given in Equation (3-90).

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c|c} T_1 & \theta \\ \hline T_2 & B_1 \end{array}\right] = \left[\begin{array}{c|c} T_1 & \theta \\ \hline T_2 & B_1 \end{array}\right] \left[\begin{array}{c|c} \theta & I \\ \hline X_1 & X_2 \end{array}\right] \tag{3-90}$$

From Equation (3-90), Equations (3-91) and (3-92) are obtained

$$A_1T_1 + A_2T_2 = \theta \tag{3-91}$$

$$A_2B_1 = T_1 \tag{3-92}$$

Writing Equations (3-91) and (3-92) in matrix form,

$$[D] \begin{array}{c} m \\ \left[\begin{array}{c} T_1 \\ \hline T_2 \end{array}\right] \end{array} \begin{array}{c} \\ m \\ m \end{array} = \begin{array}{c} m \\ \left[\begin{array}{c} \theta \\ \hline A_2B_1 \end{array}\right] \end{array} \begin{array}{c} \\ m \\ m \end{array} \tag{3-93}$$

where

$$[D] = \begin{array}{c} \begin{array}{cc} m & m \end{array} \\ \left[\begin{array}{c|c} A_1 & A_2 \\ \hline I & \theta \end{array}\right] \end{array} \begin{array}{c} \\ m \\ m \end{array}$$

In order for a unique solution to exist for T_1 and T_2 in Equation (3-93), the matrix D must have an inverse. By inspection the matrix D will have an inverse if and only if A_2 has rank = m. Since the controllability matrix C where

$$C = \left[\begin{array}{c|c} AB & B \end{array}\right] = \begin{array}{c} \begin{array}{cc} m & m \end{array} \\ \left[\begin{array}{c|c} A_2B_1 & \theta \\ \hline A_4B_1 & B_1 \end{array}\right] \end{array} \begin{array}{c} \\ m \\ m \end{array}$$

has rank = n and B_1 has rank = m then A_2 must have rank = m. Since A_2 will always have rank = m for the system to be equicontrollable with controllability index p = 2, then a solution to Equation (3-93) will always exist. Therefore the transformation matrix depends only on the B matrix and the row elements of the homogeneous states. The significance of Theorem I is that the designer now can change m rows (of the non-homogeneous states) in the system's A matrix without effecting the similarity transformation matrix. For a system with controllability

index p = 2 this means the designer has control of half the elements of his system matrix A by the use of state feedback. Example 3.4 utilizes Proposition 1 and Theorem I to solve the problem presented in Example 3.3.

Example 3.4

Given the plant and model matrices

$$A_p = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 1 \\ -1 & 2 & -1 \end{bmatrix} \qquad A_p = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$A_m = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -2 & 1 \\ 1 & -1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

First, from Proposition 1, one state is added to the system such that the augmented system is controllable with controllability index p = 2.

$$\overline{A}_p = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -10 \end{bmatrix} \tag{3-94}$$

$$\overline{B}_p = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \tag{3-95}$$

Using Curran's algorithm for generating the transformation matrix T or by solving for it directly by the method used in Theorem I, the matrix T is

$$T = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 20 & 0 & 2 \\ 0 & 2 & 0 & 0 \end{bmatrix} \tag{3-96}$$

Using the transformation matrix T, the plant is transformed to the equicontrollable canonical coordinates.

$$\overline{A}_p^{\,*} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 7 & 37 & 1 & 14 \\ 0 & -10 & 1 & -10 \end{bmatrix} \tag{3-97}$$

$$\overline{B}_p^{\,*} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3-98}$$

Now the model is augmented as the plant was.

$$\overline{A}_m = \begin{bmatrix} -1 & 2 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & -10 \end{bmatrix} \tag{3-99}$$

$$\overline{B}_m = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} . \tag{3-100}$$

Since the rows of the homogeneous states of the plant and model are equal, then by Theorem I the transformation matrix for the model will be the same as the one for the plant. The transformed model is

$$\overline{A}_m^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4.5 & 35.5 & -2.5 & 11.5 \\ .5 & -19.5 & -.5 & -12.5 \end{bmatrix} \tag{3-101}$$

$$\overline{B}_m^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{3-102}$$

The $\overline{K}^*$ is determined in the same manner as in Example 3.3

$$\overline{K}^* = \begin{bmatrix} -11.5 & -1.5 & -3.5 & -2.5 \\ .5 & -9.5 & -1.5 & -2.5 \end{bmatrix} \tag{3-103}$$

Transforming the system back to the original coordinates gives the same solution as in Example 3.3.

$$K = \begin{bmatrix} -4 & -3.5 & .5 \\ 1 & -1.5 & -.5 \end{bmatrix} \tag{3-104}$$

All of the algorithms for transforming the system equations to new coordinates have imposed the restriction that the input matrix B have full rank. If the number of inputs m is less than the order of the system, then this means the input matrix B is made up of m independent column vectors. Many systems, including the aircraft dynamics system, do not meet this restriction. Fortunately, the process can be extended to systems that do not have input matrices of full rank. Proposition 2 extends the algorithm.

Proposition 2

Given the system matrices (A,B) where A has dimension n x n and B has dimension n x r and rank m where m < r then the matrix B can be partitioned such that

$$B_{nxr} = \left[B'_{nxm} \;\middle|\; R_{nx(r-m)} \right] \tag{3-105}$$

where the submatrix B′ has rank m. The input vector $\underline{u}$ will be correspondingly partitioned

$$\underline{u} = \begin{bmatrix} \underline{u}' \\ \hline \underline{v} \end{bmatrix}$$

such that

$$B\underline{u} = B'\underline{u}' + R\underline{v}$$

And if the system matrices (A,B') meet the criteria for equicontrollability then a feedback matrix K can be determined such that

$$[sI - (A_p + B_p'K)]^{-1}\dot{B}_p = [sI - A_m]^{-1}B_m \tag{3-106}$$

To summarize what has been developed so far,

1. A system is or can be made equicontrollable if the last n columns of its controllability matrix are independent.
2. If the similarity transformations that transform two systems to equicontrollable canonical form are equal then a feedback matrix can be determined directly so that one system transfer function can be made to equal the other.
3. A necessary but not sufficient condition for the similarity transformations to be equal is that the systems' input matrices must be equal.
4. If a system is equicontrollable with a controllability index p=2 and its input matrix has n-m zero rows, then the similarity transformation matrix that will transform it to equicontrollable canonical form is dependent only on the input matrix and the row elements of the homogeneous states.

An Alternate Method

As stated previously, the problem is to find a matrix K such that

$$[sI - (A_p + B_pK)]^{-1}B_p = [sI - A_m]^{-1}B_m$$

If it is assumed that $B_p = B_m$, then the problem reduces to finding a K matrix such that

$$A_p + B_pK = A_m \tag{3-107}$$

Then,

$$B_pK = [A_m - A_p] \tag{3-108}$$

Multiplying both sides of Equation (3-108) by B_p^T

$$[B_p^T B_p]K = B_p^T[A_m - A_p] \tag{3-109}$$

If B_p has dimension n x m and rank m then $[B_p^T B_p]$ has rank m and dimension m x m, therefore $[B_p^T B_p]$ has an inverse,

$$K = [B_p^T B_p]^{-1} B_p^T [A_m - A_p] \tag{3-110}$$

On the surface Equation (3-110) looks fantastic since the only obvious restriction for determining the K matrix is that the matrix B_p have full rank. Unfortunately, however, there are more obscure restrictions. First from Equation (3-108), the rank of B_pK is less than or equal to the rank of B_p, therefore the rank of $[A_m - A_p]$ must be less than or equal to the rank of B_p. There are other structural restraints on A_m which are dependent on B_p. Substituting Equation (3-110) into Equation (3-107) gives the following necessary condition for a solution to the problem. For a particular model A_m, there is a feedback matrix K such that

$$[A_p + B_pK] = A_m$$

if and only if

$$A_p + B_p[B_p^T B_p]^{-1} B_p^T [A_m - A_p] = A_m \tag{3-111}$$

Example 3.5 reworks Example 3.3 using Equation (3-110).

Example 3.5

Given

$$A_p = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 1 \\ -1 & 2 & -1 \end{bmatrix} \qquad B_p = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$A_m = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -2 & 1 \\ 1 & -1 & -2 \end{bmatrix} \qquad B_m = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$B_p^T B_p = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

$$[B_p^T B_p]^{-1} B_p^T = \begin{bmatrix} 0 & 1 & -.5 \\ 0 & 0 & .5 \end{bmatrix}$$

Using Equation (3-111)

$$A_p + B_p[B_p^T B_p]^{-1} B_p^T [A_m - A_p] = A_m$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 1 \\ -1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ -3 & -5 & 0 \\ 2 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ -1 & -2 & 1 \\ 1 & -1 & -2 \end{bmatrix} = A_m \tag{3-112}$$

Equation (3-112) shows that Equation (3-111) is satisfied, therefore there is a solution. This solution is

$$K = [B_p{}^T B_p]^{-1} B_p{}^T [A_m - A_p] = \begin{bmatrix} -4 & -3.5 & .5 \\ 1 & -1.5 & -.5 \end{bmatrix}$$

Again the solution is the same as that found in Examples 3.3 and 3.4. It should be restated that there is not always a solution to the problem via Equation (3-110) and to insure that a solution does exist Equation (3-111) should be checked.

This process can be extended to systems where the input matrix of the model does not equal that of the plant.

<u>Proposition 3</u>

Given

$$\dot{\underline{x}}_p = A_p \underline{x}_p + B_p \underline{u}$$

and

$$\dot{\underline{x}}_m = A_m \underline{x}_m + B_m \underline{u}$$

where A_p and A_m have dimension n x n, B_p and B_m have dimension m x m, and B_p has full rank then there is a K matrix and a Q matrix such that

$$[sI - (A_p + B_p K)]^{-1} B_p Q = [sI - A_m]^{-1} B_m \qquad (3\text{-}114)$$

if and only if

$$A_p + B_p [B_p{}^T B_p]^{-1} B_p{}^T [A_m - A_p] = A_m \qquad (3\text{-}115)$$

and

$$B_p[B_p^T B_p]^{-1} B_p^T B_m = B_m \tag{3-116}$$

If Equations (3-115) and (3-116) are satisfied then,

$$K = [B_p^T B_p]^{-1} B_p^T [A_m - A_p] \tag{3-117}$$

and

$$Q = [B_p^T B_p]^{-1} B_p^T B_m \tag{3-118}$$

The block diagram for the compensated system is given in Figure 3-9.

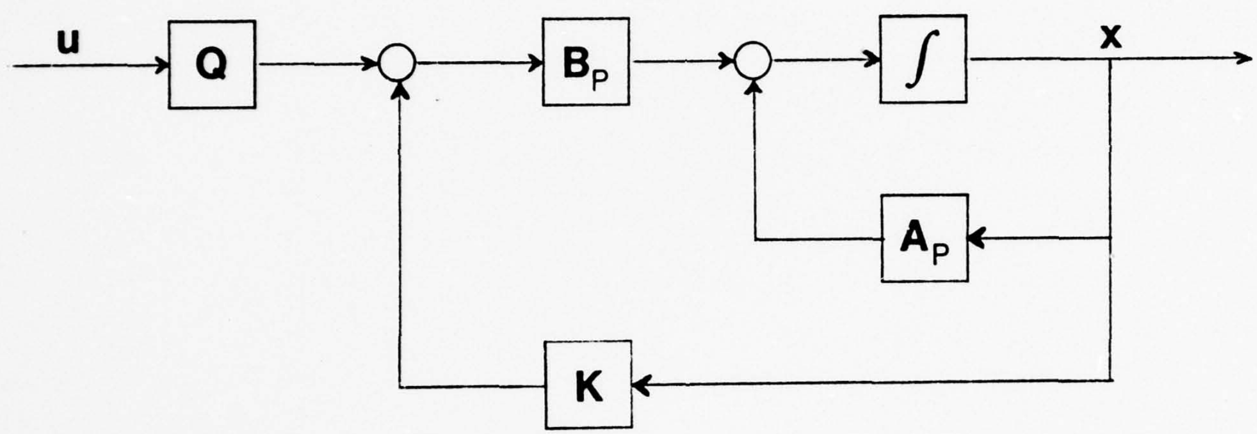


Figure 3-9. Block Diagram for Compensated System Given in Proposition 3.

IV. APPLICATION OF COMPENSATOR DESIGN ALGORITHMS TO THE AIRCRAFT LONGITUDINAL CONTROL PROBLEM

As shown in Equation (2-17) the aircraft longitudinal system in state space form has an A matrix with dimension 5 x 5 and a B matrix with dimension 5 x 4 where the matrices are given in Figures 2-2 and 2-3 respectively. Since the B matrix has two rows of zeroes it has rank less than or equal to 3 and is not of full rank. An inspection of the B matrix reveals that the first, second, and third columns are linearly independent; therefore the B matrix has rank = 3. Since it is necessary for the input matrix to have full rank, as shown in Proposition 2, one input will be neglected so that the modified B matrix will have full rank. After the compensator is designed, the neglected input will be added back into the system. Since the first three columns of the B matrix are linearly independent, $U_4 = \Delta C_{D_c}$ will be neglected. The modified system now is of the form

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

where

A has dimension 5 x 5

B has dimension 5 x 3

The $\underline{x}$ and $\underline{u}$ vectors are now

$$\underline{x} = \begin{bmatrix} \Delta V \\ \Delta\alpha \\ \theta \\ q \\ \Delta Z \end{bmatrix} \qquad (4\text{-}1)$$

$$\underline{u} = \begin{bmatrix} \Delta C_{T_c} \\ \Delta C_{L_c} \\ \Delta C_{m_c} \end{bmatrix} \qquad (4\text{-}2)$$

and the new B matrix will be the first three columns of the original B matrix.

If the system is controllable with controllability index p = 2, then one state will have to be added to the system to make it equicontrollable (N=n+1=pm). The dimension of the augmented system will be N = 6.

The augmented system input matrix has N-m = 3 zero rows, two from the original system and one from the additional state. Theorem I showed that if the system had a controllability index p = 2 and n - m rows of zeroes in the B matrix, then the similarity transformation matrix T is dependent only on the B matrix and the row elements of the homogeneous states. Therefore the model must have the same row elements as the plant in the third and fifth rows of its A matrix and the additional state must be added in the same manner for both systems. This is no problem since it was shown in Chapter II that the desired response could

be obtained by adjusting the elements of the first, second, and fourth rows of the A matrix.

Thus far it has been shown that the modified system

1. has a B matrix with full rank = 3 and $n - m$ zero rows.
2. can be altered to obtain the desired response without effecting the similarity transformation matrix.
3. will have to be augmented with one additional state to make it equicontrollable.

Now the only criterion remaining to be fulfilled to insure a solution to the problem is that the system must be controllable with controllability index $p = 2$. Controllability will be dependent on the values of the A and B matrix elements or more precisely on the aircraft and its aerodynamic coefficients.

Examples 4.1 and 4.2 illustrate the algorithms that have been presented using data on two different aircraft. A computer program was written to compute the A and B matrices from the general longitudinal aircraft equations and the aerodynamic coefficients of the aircraft in question. The program computes the system transfer function, the roots of the characteristic equation, and the short period and phugoid damping ratios and natural frequencies. The program was written in FORTRAN IV and run on the IBM-370 system using double precision arithmetic.

Example 4.1

Using data on a business jet aircraft, the A and B matrices were determined. They are

$$A_p = \begin{bmatrix} -7.54(10)^{-3} & 8.47 & -32.2 & 0 & 0 \\ -2.08(10)^{-4} & -.669 & 0 & .996 & -4.96(10)^{-6} \\ 0 & 0 & 0 & 1 & 0 \\ 9.47(10)^{-4} & -7.18 & 0 & -1.35 & 2.02(10)^{-6} \\ 0 & 675 & -675 & 0 & 0 \end{bmatrix} \tag{4-3}$$

$$B_p = \begin{bmatrix} 76.89 & 0 & 0 & -76.98 \\ -5.36(10)^{-3} & -.114 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2.18(10)^{-3} & .046 & 11.64 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4-4}$$

The characteristic equation for the system is

$$\text{C.E.} = \lambda^5 + 2.02\lambda^4 + 8.07\lambda^3 + .0901\lambda^2 + .0935\lambda + 2.57(10)^{-4} = 0 \tag{4-5}$$

The roots of the characteristic equation are

$$\lambda_1 = -2.75(10)^{-3}$$
$$\lambda_2, \lambda_3 = -2.77(10)^{-3} \pm i(.108)$$
$$\lambda_4, \lambda_5 = -1.01 \pm i(2.65)$$

From the characteristic roots the phugoid and short period damping

ratios and natural frequencies were determined. They are

$$\zeta_p = .026 \quad , \quad \omega_{n_p} = .108 \qquad (4\text{-}6)$$

$$\zeta_{sp} = .355 \quad , \quad \omega_{n_{sp}} = 2.835 \qquad (4\text{-}7)$$

N_α and L_α/mV were computed using

$$L_\alpha/mV = 1/2\rho V^2 S C_{L_\alpha}/mV \qquad (4\text{-}8)$$

and

$$N_\alpha = L_\alpha/mg \qquad (4\text{-}9)$$

They are

$$L_\alpha/mV = .666$$

$$N_\alpha = 13.9$$

Since $N_\alpha < 15$, the performance criteria given in Figure 2-4 will be used to evaluate the short response of the open loop plant and to determine the desired $\omega_{n_{sp}}$ and ζ_{sp}. Figure 4-1 shows the location of the response characteristics of the uncompensated system and the desired system on the performance criteria plot.

From the desired location on the performance plot given in Figure 4-1, $\omega_{n_{sp}}$ and ζ_{sp} for the model can be determined to be

$$\omega_{n_{sp}} = 1.5 \qquad (4\text{-}10)$$

$$\zeta_{sp} = .7 \qquad (4\text{-}11)$$

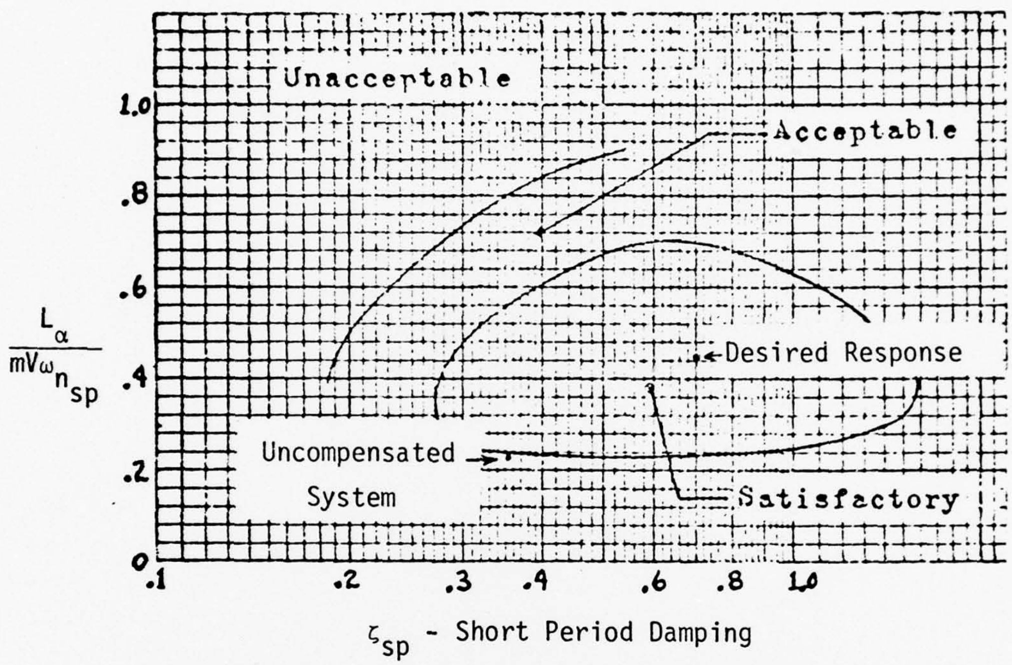


Figure 4-1. Handling Qualities For System in Example 4.1.

Now the desired response for the phugoid mode must be determined. From Equation (4-6) it is noted that the phugoid period is approximately 60 seconds with very little phugoid damping. From the performance criteria it was shown that a minimum damping of .2 is desired. In the model ω_{n_p} will be decreased and ζ_p will be increased to approximately .4.

To develop the model a_{22}, a_{24}, a_{42}, and a_{44} were selected in the manner given in Chapter II. Using the values of ζ_{sp} and $\omega_{n_{sp}}$ given in Equations (4-10) and (4-11) the desired short period characteristic equation was determined. It is

$$C.E._{sp} = \lambda^2 + 2.1\lambda + 2.25 \tag{4-12}$$

First a_{22_m} is set equal to a_{22_p} and a_{24} is set equal to 1. Therefore

$$a_{22_m} = .669$$

$$a_{24_m} = 1.$$

Using the value of a_{22_m} and the fact that $a_{24_m} = 1$, a_{42_m} and a_{44_m} are determined by

$$a_{44_m} = -(a_{22_m} + 2\zeta_{sp}\omega_{n_{sp}}) = .669 - 2.1 = -1.43$$

and

$$a_{42_m} = (a_{22_m})(a_{44_m}) - \omega_{n_{sp}}^2 = .96 - 2.25 = -1.29$$

Adjusting the phugoid roots is considerably more difficult than adjusting the short period roots as shown in Chapter II. As a general rule of thumb, increasing the magnitude of the a_{11} element increases the phugoid damping and decreasing the magnitude of the a_{12}, a_{13}, or a_{21} element decreases the phugoid natural frequency. The phugoid approximation given in Chapter II seems to break down as the phugoid damping becomes large.

After several trys, a satisfactory model was determined. The model A matrix is given in Equation (4-13).

$$A_m = \begin{bmatrix} -.1 & 8.47 & -20.0 & 0 & 0 \\ -(10)^{-4} & -.669 & 0 & 1 & -4.96(10)^{-6} \\ 0 & 0 & 0 & 1 & 0 \\ 9.47(10)^{-4} & -1.29 & 0 & -1.43 & 2.02(10)^{-6} \\ 0 & 675 & -675 & 0 & 0 \end{bmatrix} \tag{4-13}$$

The model characteristic equation is

$$C.E. = \lambda^5 + 2.2\lambda^4 + 2.46\lambda^3 + .242\lambda^2 + .021\lambda + 5.6(10)^{-4} = 0$$

and its pole locations are

$$\lambda_1 = -.037$$
$$\lambda_2,\lambda_3 = -.031 \pm i(-.076)$$
$$\lambda_4,\lambda_5 = -1.05 \pm i(1.07)$$

and the damping ratios and natural frequencies are

$$\zeta_p = .38 \quad , \quad \omega_{n_p} = .0825 \; .$$

$$\zeta_{sp} = .7 \quad , \quad \omega_{n_{sp}} = 1.499$$

B_m was made the same as B_p to insure the similarity transformations were the same between the plant and model.

Now that a satisfactory model has been constructed, a compensator which will cause the aircraft to respond in the same manner as the model will be designed. First the compensator will be designed using Curran's method. The compensator was designed on the HP-2000 timeshare computer using Curran's algorithm as outlined in Chapter III.

First the plant was augmented and placed in equicontrollable canonical form. One state was added to the system to make it equicontrollable by the process given in Proposition 1. The additional state equation is

$$\dot{x}_6 = x_1 - 10x_6 \tag{4-14}$$

The augmented system is now of the form

$$\hat{\dot{\underline{x}}} = \hat{A}\hat{\underline{x}} + \hat{B}\underline{u} \tag{4-15}$$

where

$$\hat{A} = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & 0 \\ & & A & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & -10 \end{array}\right] \tag{4-16}$$

and

$$\hat{B} = \left[\begin{array}{ccc} & B & \\ \hdashline 0 & 0 & 0 \end{array}\right] \tag{4-17}$$

In order to transform the system to equicontrollable canonical form, transformation matrix T_p was generated. It is

$$T_p = \begin{bmatrix} 768.9 & 0 & 0 & 76.9 & 0 & 0 \\ 2.2(10)^{-3} & .046 & 11.64 & -5.4(10)^{-3} & -.114 & 0 \\ 2.2(10)^{-3} & .046 & 11.64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.2(10)^{-3} & .046 & 11.64 \\ -3.62 & -76.9 & 0 & 0 & 0 & 0 \\ 76.9 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4-18}$$

The plant matrices in equicontrollable canonical form are

$$\hat{A}_p^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -.076 & -.014 & -3.60 & -10.0 & -.013 & 0 \\ 1.42 & .270 & 68.5 & .58 & -.66 & .419 \\ .055 & -.030 & -7.45 & .009 & .068 & -1.35 \end{bmatrix} \tag{4-19}$$

and

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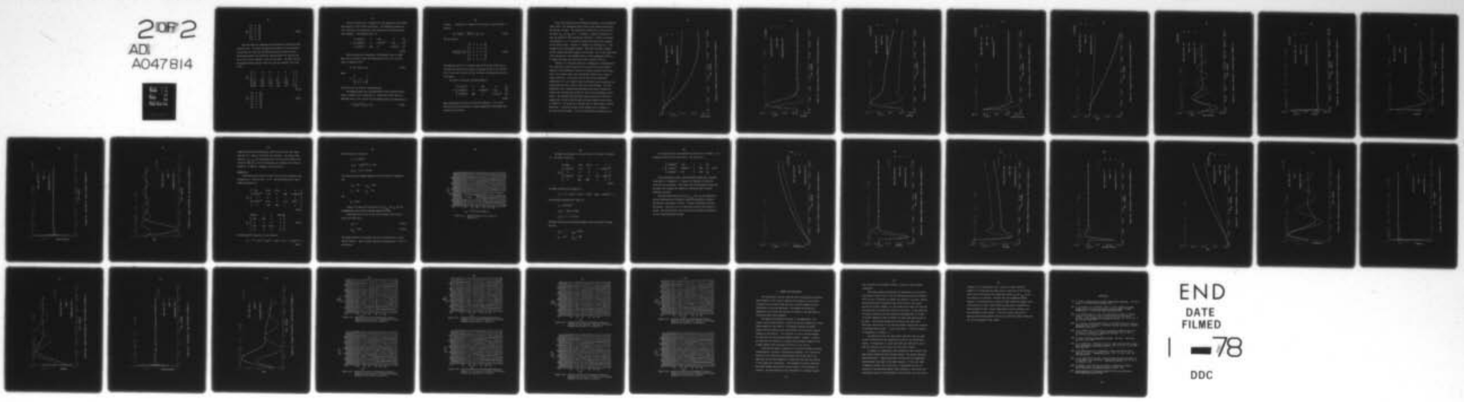

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$$\hat{B}_p^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \qquad (4\text{-}20)$$

Next the model was augmented and transformed to equicontrollable canonical form. The state was added to the model in the same manner as the plant to insure the transformation matrices would be equal. The program compares the similarity transformations for equality and exits with an error message if they are not equal. The model matrices in equicontrollable canonical form are given by Equations (4-21) and (4-22).

$$\hat{A}_m^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1.0 & -.007 & -1.75 & -10.1 & -.013 & 0 \\ .735 & .269 & 68.4 & .512 & -.668 & 0 \\ .060 & -.006 & -1.56 & .006 & .010 & -1.43 \end{bmatrix}$$

(4-21)

$$\hat{B}_m^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4\text{-}22)$$

Using the method given in Chapter III the compensator was designed and returned to the original coordinates. The compensator output was not a function of the additional state therefore it consisted of pure state feedback. The feedback matrix is

$$K = \begin{bmatrix} -1.2(10)^{-3} & 0 & .159 & 0 & 0 \\ -8.93(10)^{-4} & 0 & -7.47(10)^{-3} & -.036 & 0 \\ 3.78(10)^{-6} & .505 & 0 & -7.2(10)^{-3} & 0 \end{bmatrix} \tag{4-23}$$

After designing the compensator, the neglected input is added back into the system so that the compensated system is of the form given in Equation (4-24).

$$\dot{\underline{x}} = [A + BK']\underline{x} + B\underline{u} \tag{4-24}$$

where

$$K' = \left[\begin{array}{ccccc} & & K & & \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and A and B are the original system matrices.

The feedback matrix was also determined by the alternate method given in Chapter III by Proposition 3. Using the matrices given in Equations (4-3), (4-4), and (4-13) the feedback matrix was determined by

$$K = [B_p{}^T B_p]^{-1} B_p{}^T [A_m - A_p] \tag{4-25}$$

As shown in Proposition 3, Equation (4-25) gives a valid solution if and only if

$$A_p + B_p[B_p^T B_p]^{-1} B_p^T [A_m - A_p] = A_m \tag{4-26}$$

For this system

$$B_p[B_p^T B_p]^{-1} B_p^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4-27}$$

From Equation (4-27) it is obvious that the third and fifth rows of the plant and model must be equal for Equation (4-26) to be satisfied. This is the same criterion for the similarity transformation matrices to be equal.

The K matrix using the alternate method is

$$K = \begin{bmatrix} -1.2(10)^{-3} & 0 & .159 & 0 & 0 \\ -8.93(10)^{-4} & 0 & -7.49(10)^{-3} & -.036 & 0 \\ 3.78(10)^{-6} & .505 & 0 & -7.2(10)^{-3} & 0 \end{bmatrix} \tag{4-28}$$

Note that Equations (4-28) and (4-23) are identical. This is not surprising since the solution is unique regardless of the method used to obtain the solution.

Using CSMP (Continuous System Modeling Program), the uncompensated plant, model, and compensated plant impulse time response curves were plotted and compared. The systems were perturbed by a delta function on inputs ΔC_{L_c} and ΔC_{m_c} at t = .5 seconds. Figures 4-2 through 4-11 give the results of the simultaneous simulation. Figures 4-2 through 4-6 terminate at t = 15 seconds to compare the short period response of the system states. Figures 4-7 through 4-11 terminate at t = 250 seconds to show the phugoid response. Note that the model response and the compensated plant response are the same. This is not surprising since the purpose of the feedback matrix in the compensated system is to equate the model and compensated system transfer functions.

Example 4.2 illustrates that once a compensator is designed which will cause the aircraft response to be well within the satisfactory regions of the performance criteria at a median velocity and altitude, then it will remain within that satisfactory region over a range of flight conditions. Wind tunnel data of the various aerodynamic coefficients for a jet fighter type aircraft were used to construct the system matrices over a range of velocities and altitudes. From the aerodynamic data, interpolating polynomials that were a function of velocity and altitude were written for each of the aerodynamic coefficients. The interpolating polynomials were accurate for velocities ranging from 330 fps to 880 fps and altitudes ranging from sea level to 30,000 ft. The system was simulated only in these ranges of flight conditions. A modified version of the program used in Example 4.1 was used for this program. Using the interpolating polynomials, the

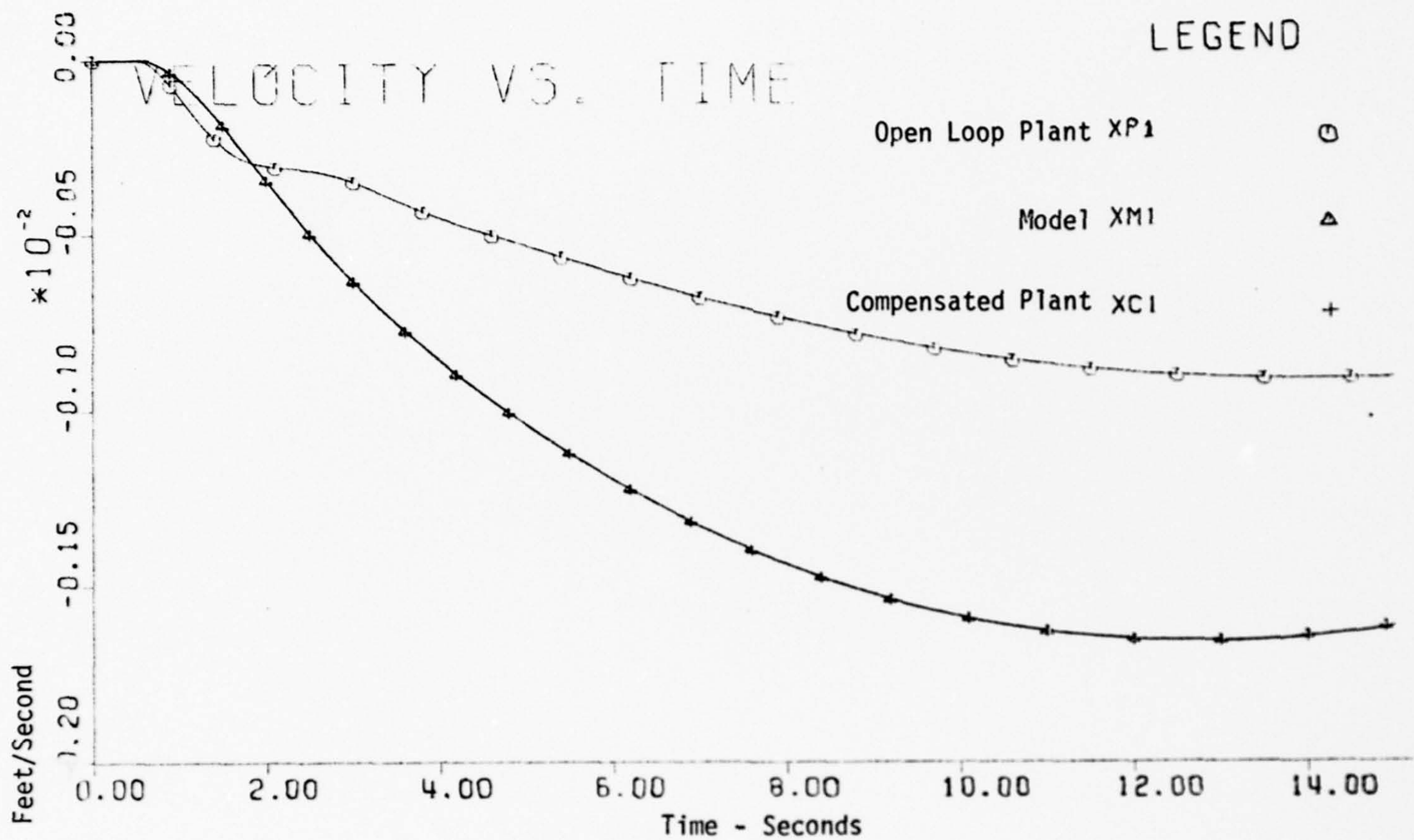

Figure 4-2. Change in Velocity vs. Time For System in Example 4.1.

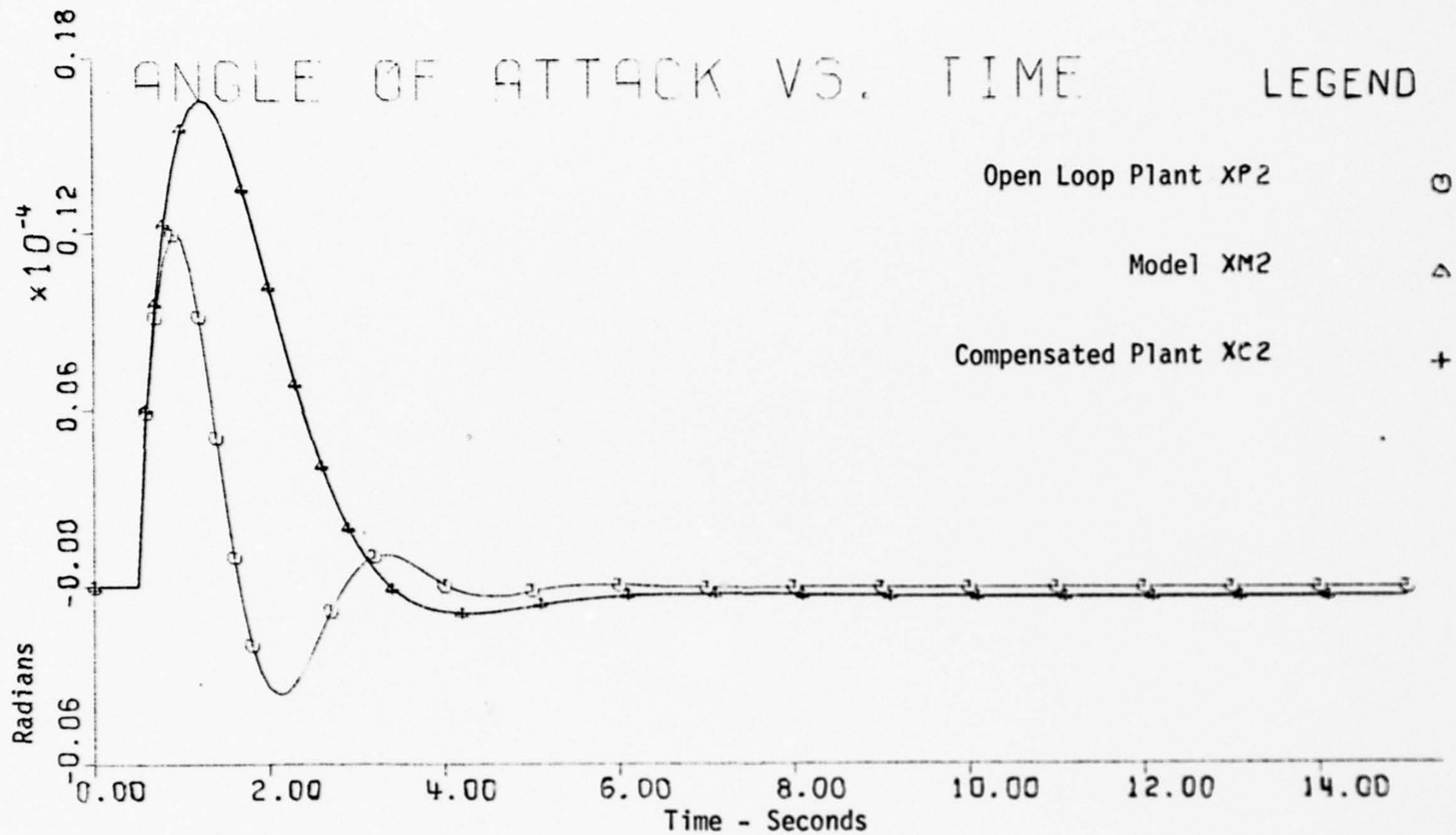

Figure 4-3. Change in Angle of Attack vs. Time For System in Example 4.1.

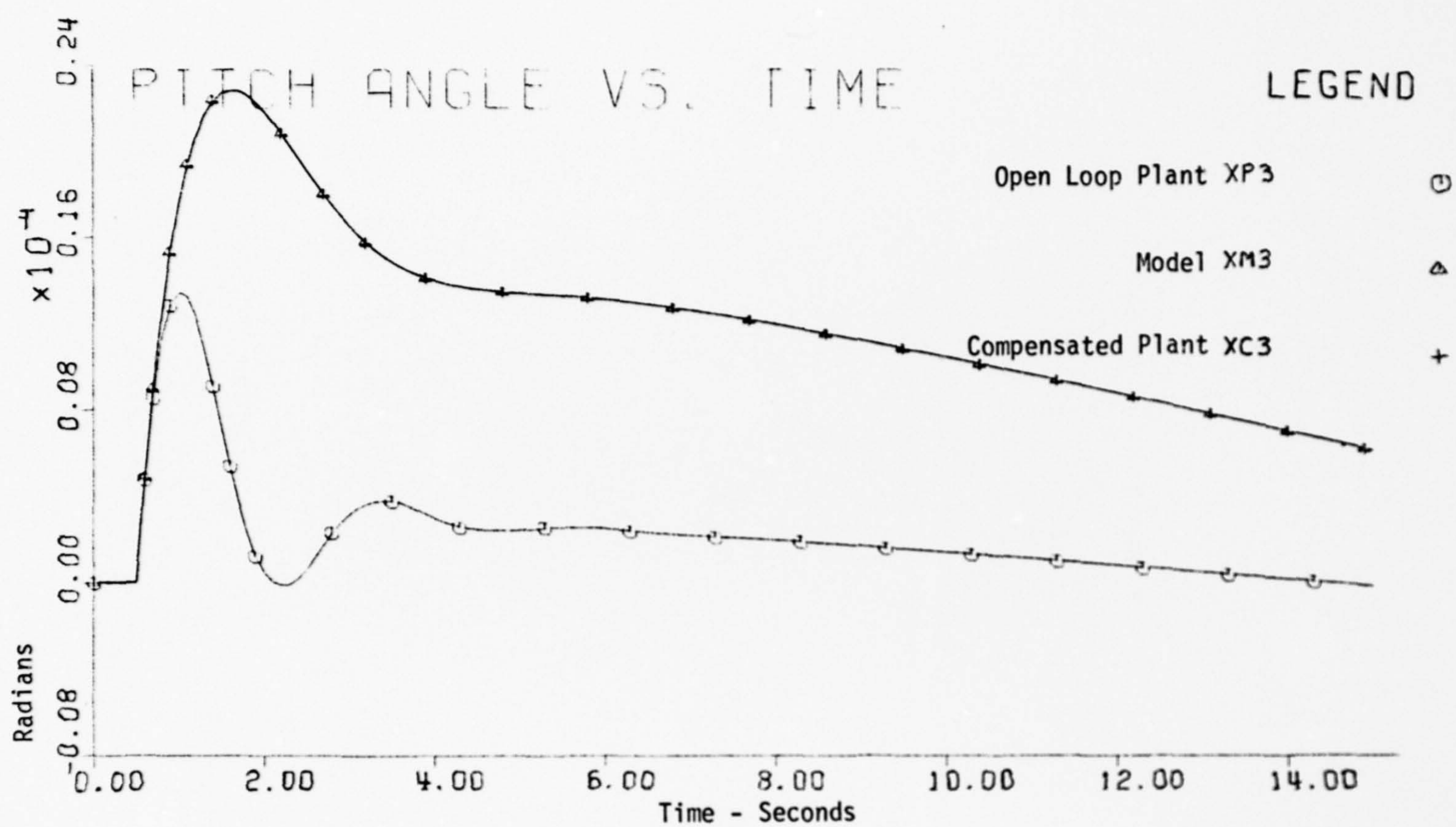

Figure 4-4. Change in Pitch Angle vs. Time For System in Example 4.1.

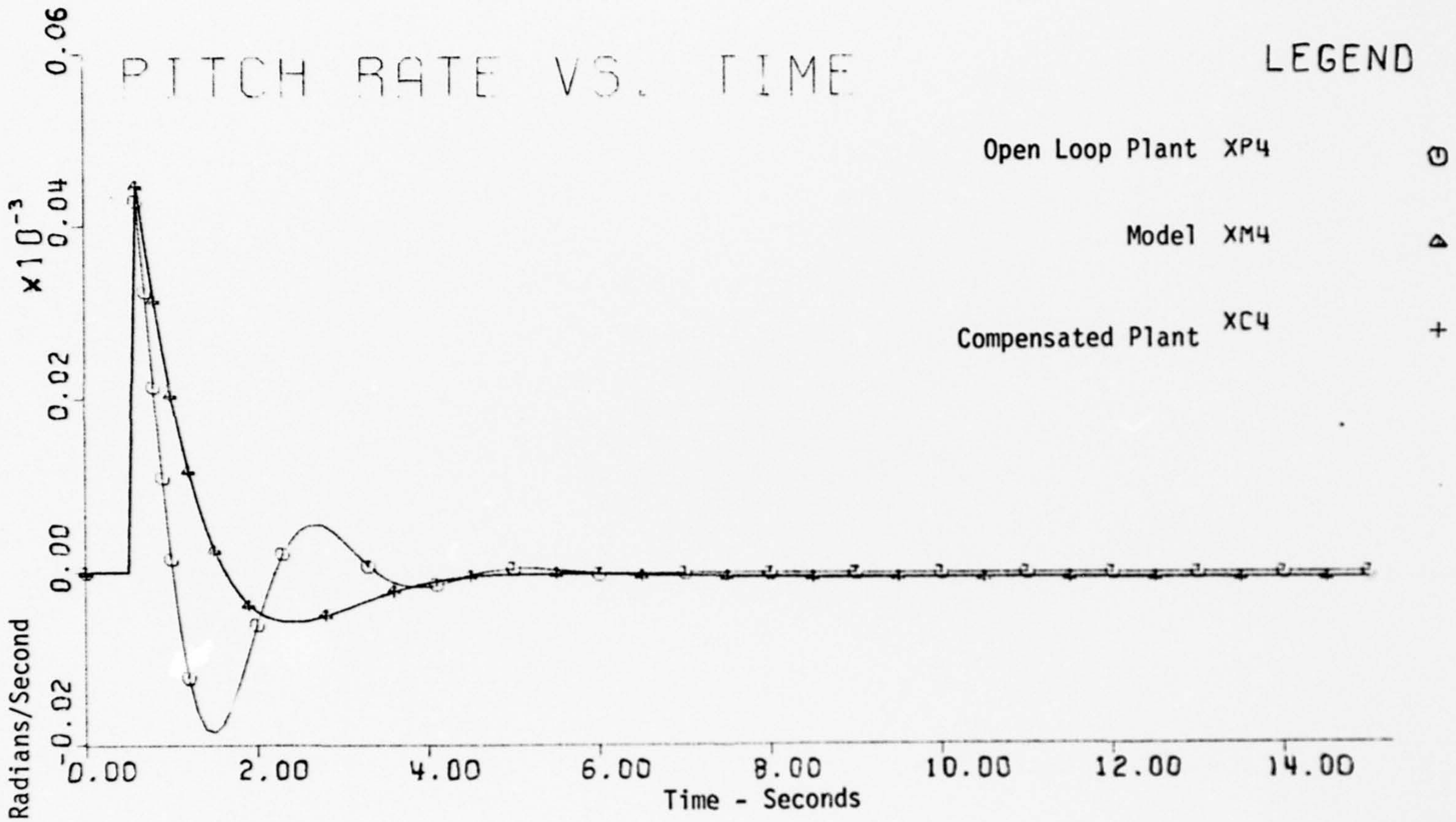

Figure 4-5. Pitch Rate vs. Time For System in Example 4.1.

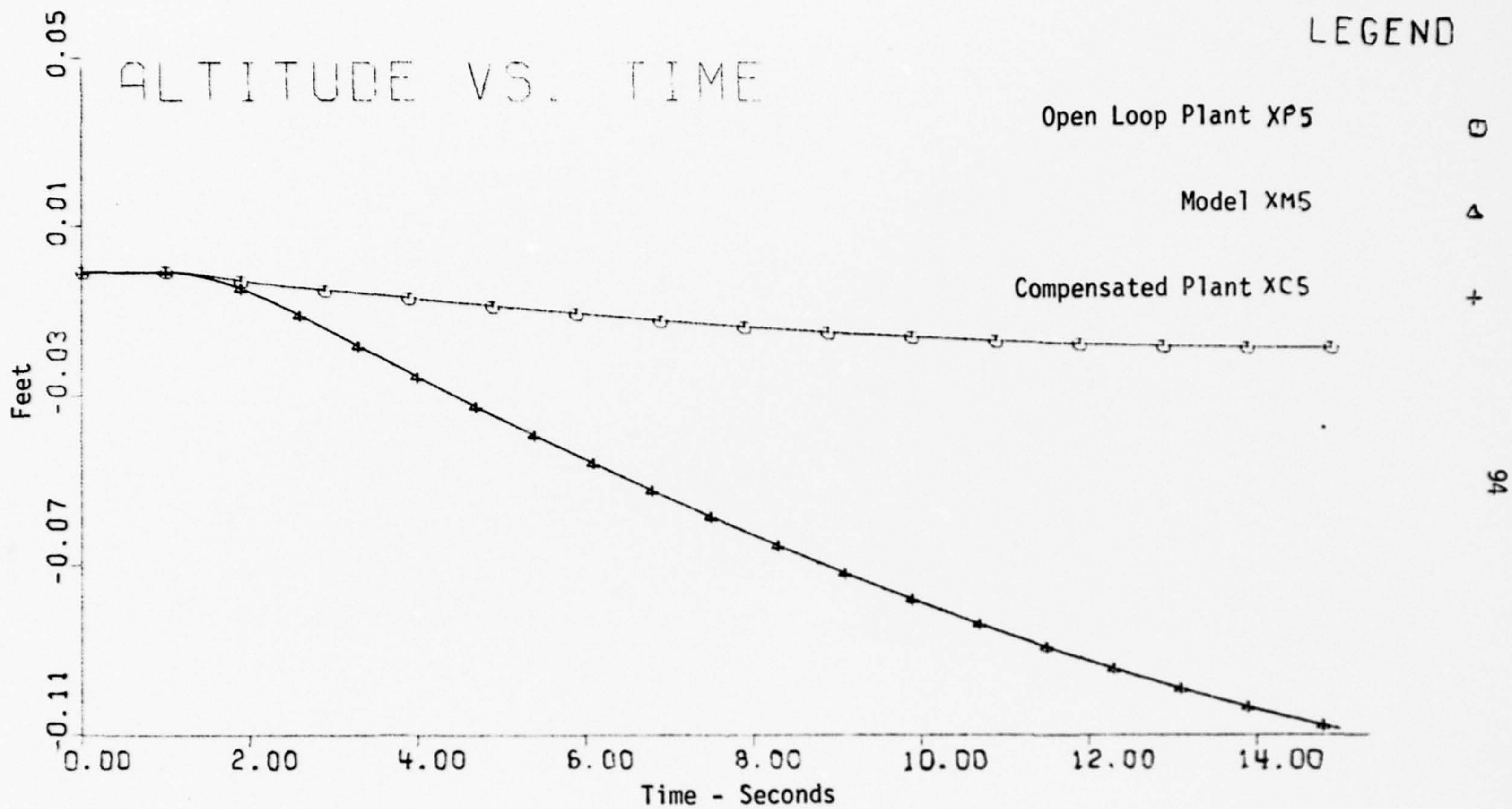

Figure 4-6. Change in Altitude vs. Time For System in Example 4.1.

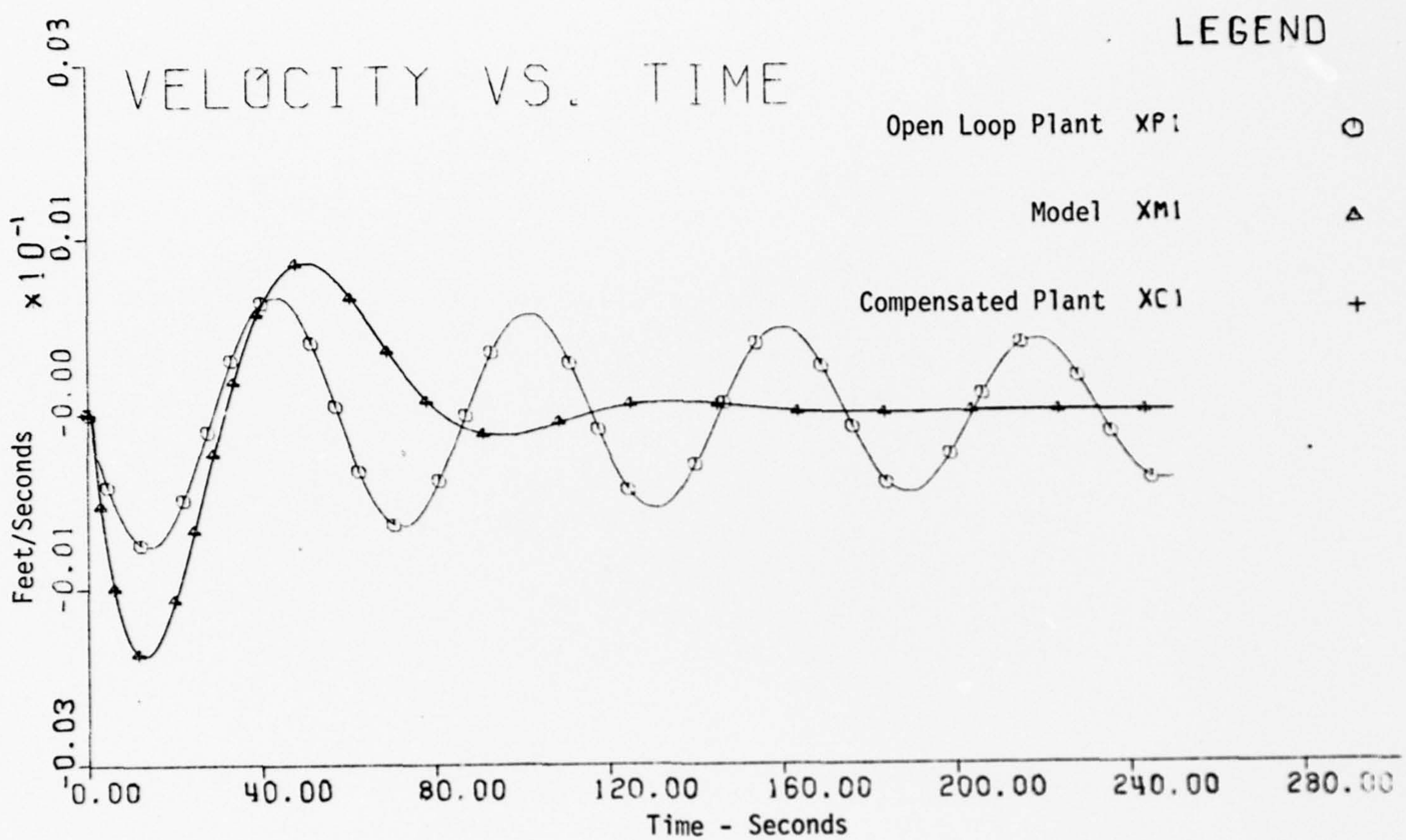

Figure 4-7. Change in Velocity vs. Time For System in Example 4.1.

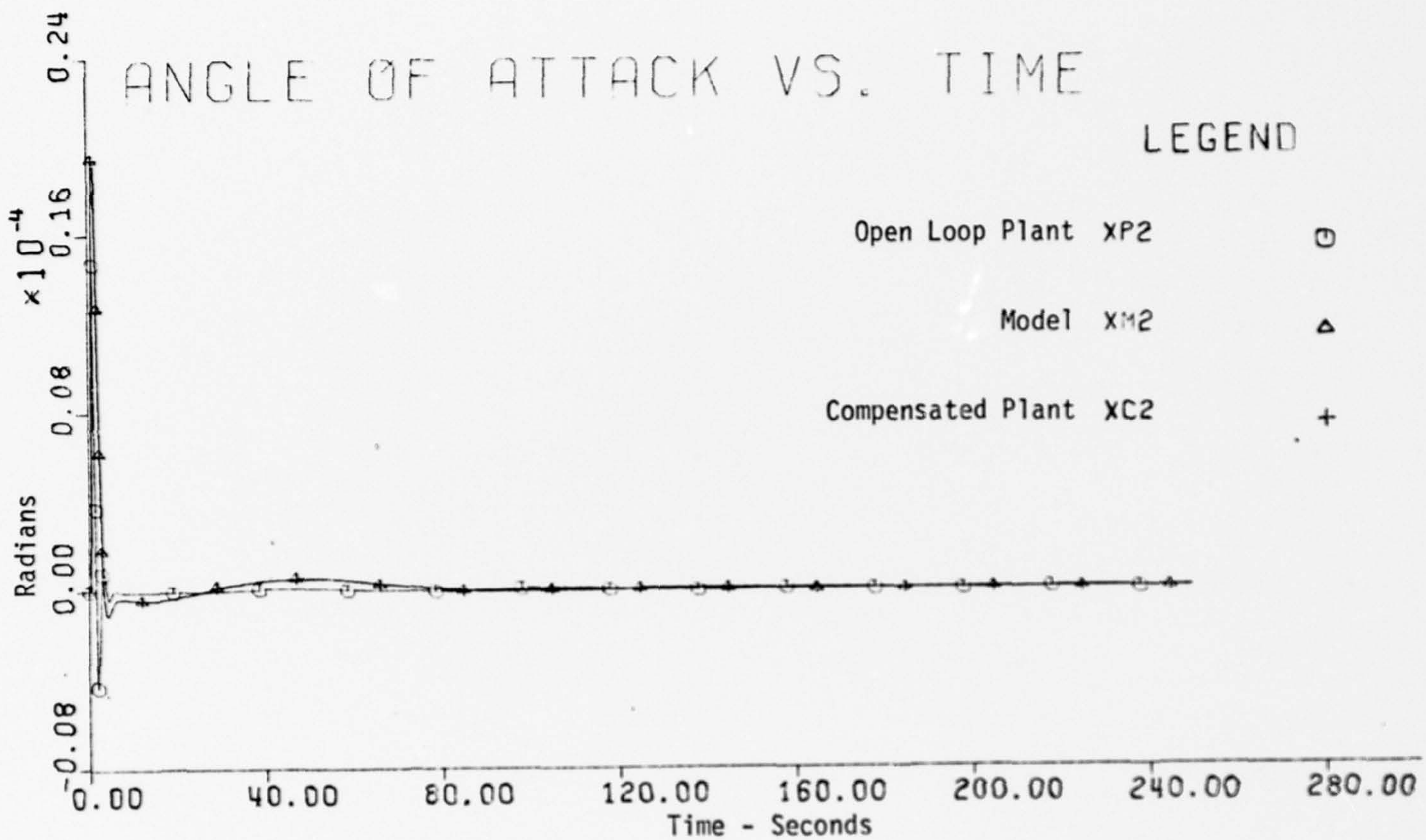

Figure 4-8. Change in Angle of Attack vs. Time For System in Example 4.1.

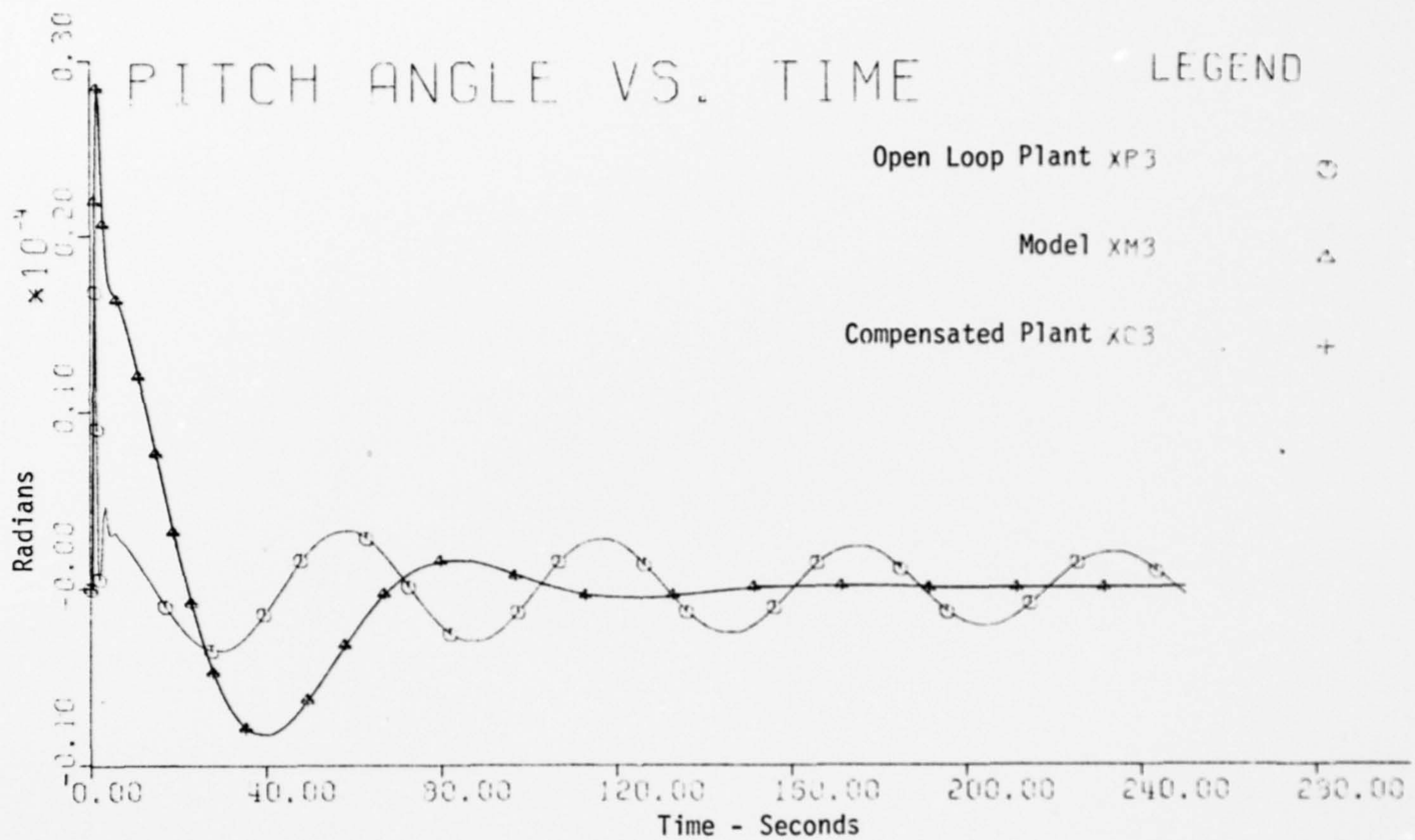

Figure 4-9. Change in Pitch Angle vs. Time For System in Example 4.1.

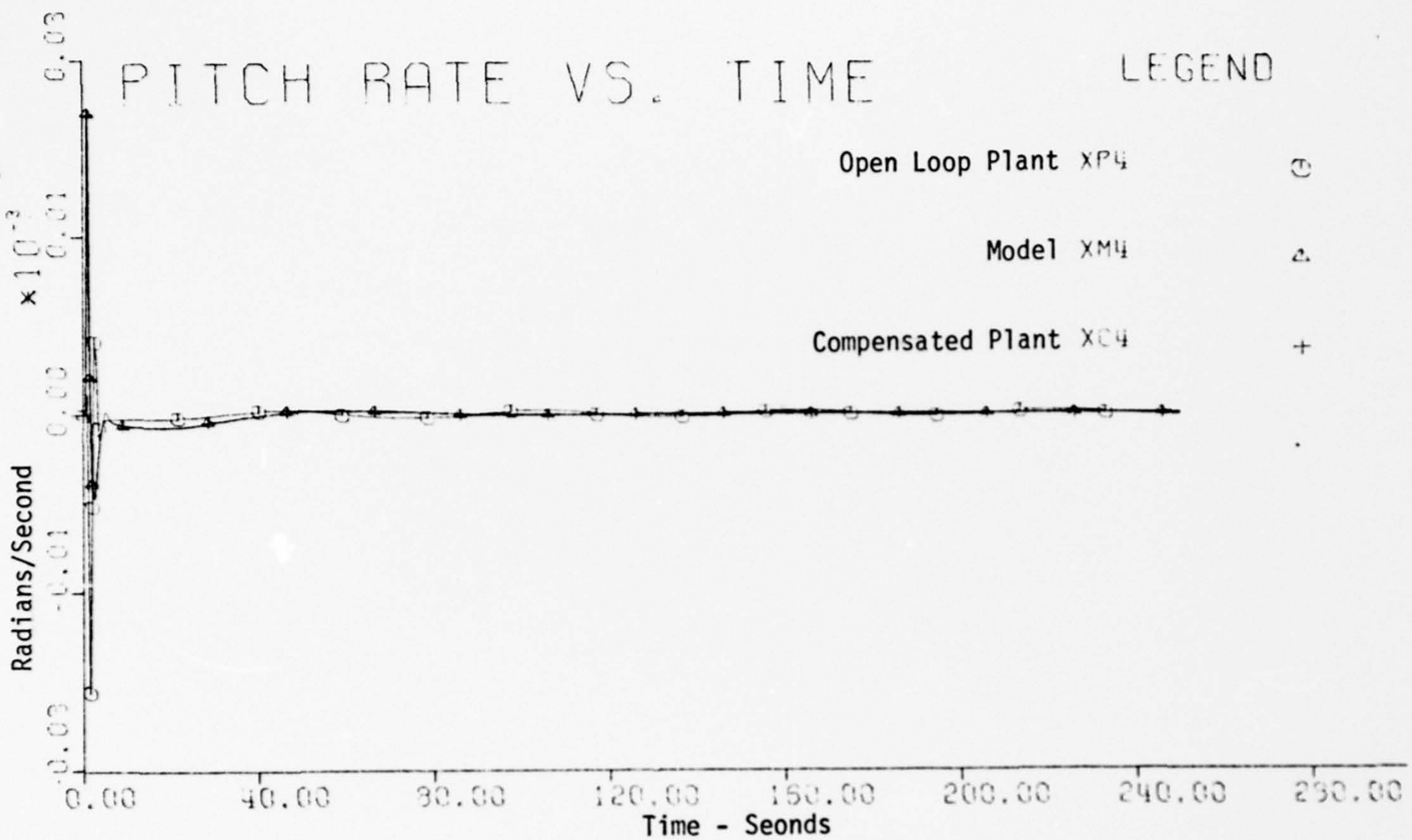

Figure 4-10. Pitch Rate vs. Time For System in Example 4.1.

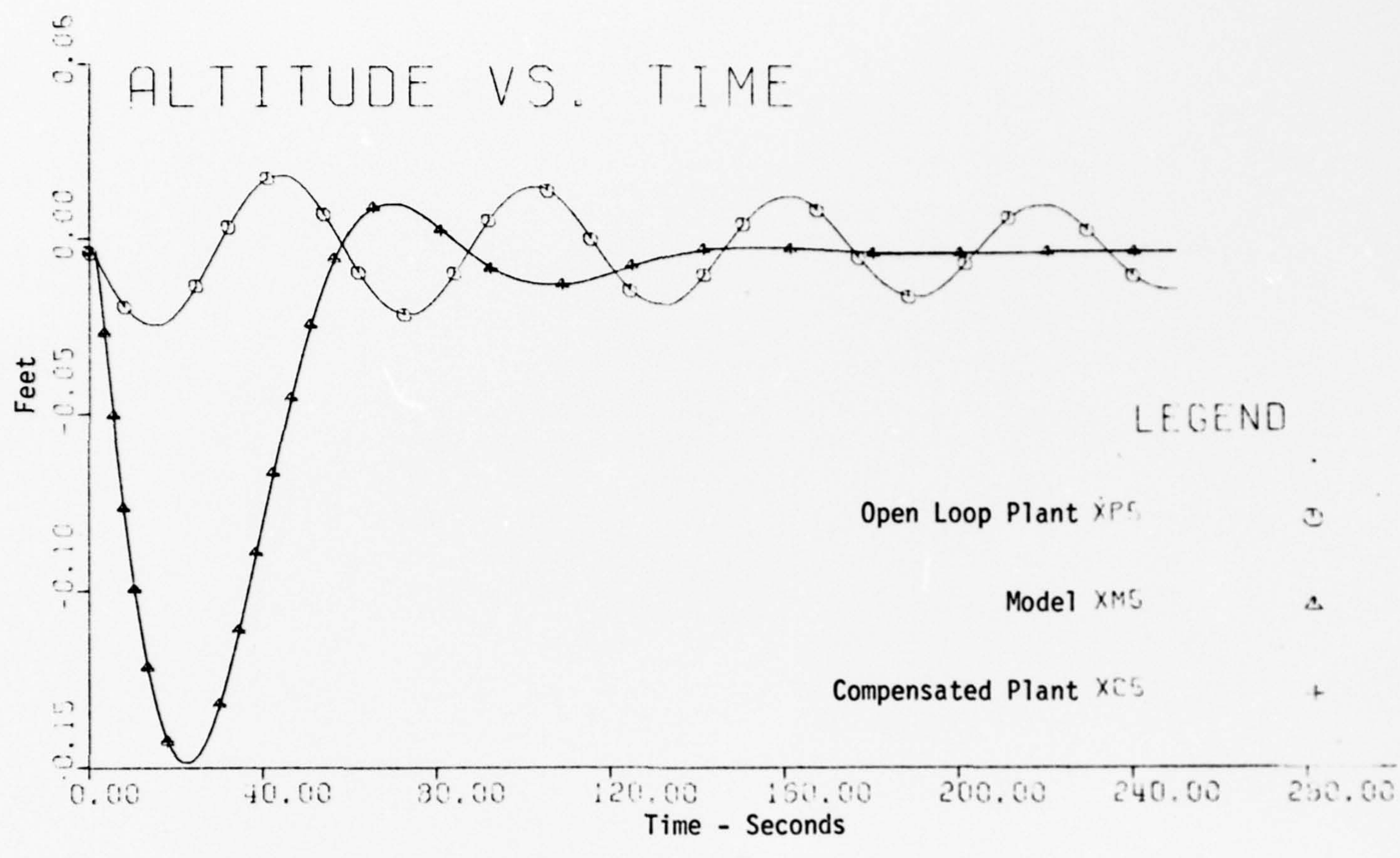

Figure 4-11. Change in Altitude vs. Time For System in Example 4.1.

program calculated the aerodynamic coefficients and then the system matrices for a range of velocities and altitudes. The program tabulated N_α, ζ_{sp}, ω_{sp}, and the phugoid roots for velocities ranging from 330 fps to 880 fps in 110 fps increments and altitudes from 0 feet to 30,000 ft. in 5000 ft. increments at each velocity.

Example 4.2

The velocity and altitude at which this aircraft compensator was designed was V = 660 fps and Z = 0 ft. The system matrices at these flight conditions are

$$A_p = \begin{bmatrix} -.016 & 16.45 & -32.15 & 0 & 0 \\ -1.45(10)^{-4} & -1.19 & 0 & .988 & -1.26(10)^{-6} \\ 0 & 0 & 0 & 1 & 0 \\ 5.75(10)^{-5} & -8.10 & 0 & -1.22 & 5(10)^{-7} \\ 0 & 660 & -660 & 0 & 0 \end{bmatrix} \tag{4-29}$$

$$B_p = \begin{bmatrix} 257.94 & 0 & 0 & -259.03 \\ -.036 & -.392 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ .014 & .155 & 34.29 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4-30}$$

The characteristic equation for the system is

$$C.E. = \lambda^5 + 2.43\lambda^4 + 9.50\lambda^3 + .156\lambda^2 + .047\lambda + 1.14(10)^{-4} = 0 \tag{4-31}$$

and the roots are located at

$$\lambda_1 = -2.44(10)^{-3}$$

$$\lambda_2, \lambda_3 = -6.38(10^{-3}) \pm i(.07)$$

$$\lambda_4, \lambda_5 = -1.21 \pm i(2.83)$$

The short period and phugoid damping ratios and natural frequencies are

$$\zeta_{sp} = .39 \quad , \quad \omega_{n_{sp}} = 3.08$$

$$\zeta_p = .09 \qquad \omega_{n_p} = .07$$

and

$$N_\alpha = 24.27$$

Figure 4-12 shows the intersection of $N_\alpha/\omega_{n_{sp}}$ and ζ_{sp} for the uncompensated plant and the proposed compensated plant.

From Figure 4-12 it can be seen that the model short period ζ and ω_n will have to be

$$\zeta_{sp} = .7 \tag{4-32}$$

$$\omega_{n_{sp}} = 2.427 \tag{4-33}$$

The phugoid damping for the model must also be increased for a satisfactory response. Again a phugoid damping of approximately .4 will be satisfactory.

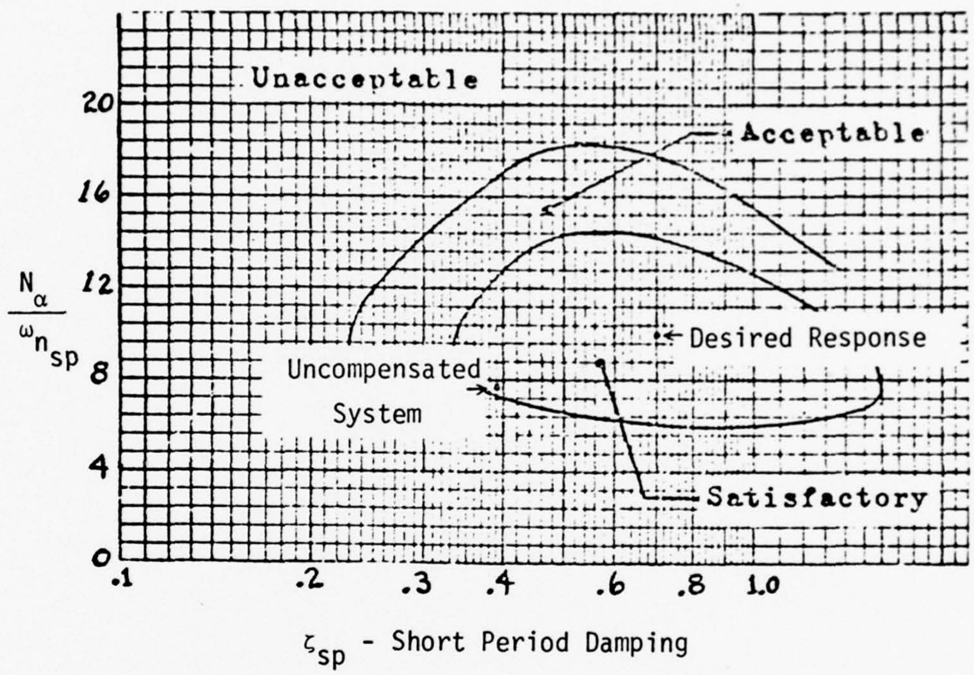

Figure 4-12. Handling Qualities For System In Example 4.2.

The model was designed in the same manner as the model in Example 4.1. The model A matrix is

$$A_m = \begin{bmatrix} -.0625 & 3.30 & -32.15 & 0 & 0 \\ -1.83(10)^{-4} & -1.19 & 0 & 1 & -1.26(10)^{-6} \\ 0 & 0 & 0 & 1 & 0 \\ 5.75(10)^{-5} & -3.27 & 0 & -2.21 & 5(10)^{-7} \\ 0 & 660 & -660 & 0 & 0 \end{bmatrix} \tag{4-34}$$

The model characteristic equation is

$$\text{C.E.} = \lambda^5 + 3.46\lambda^4 + 6.10\lambda^3 + .373\lambda^2 + .025\lambda + 1.95(10)^{-4} = 0$$

and the model characteristic roots are

$$\lambda_1 = -8.91(10)^{-3}$$

$$\lambda_2, \lambda_3 = -.026 \pm i(.055)$$

$$\lambda_4, \lambda_5 = -1.7 \pm i(1.73)$$

The model short period and phugoid damping ratios and natural frequencies are

$$\zeta_{sp} = .7 \quad , \quad \omega_{n_{sp}} = 2.427$$

$$\zeta_p = .43 \qquad \omega_{n_{sp}} = .061$$

The feedback matrix was determined by both Curran's method and the alternate method with the same result. The solution is

$$K = \begin{bmatrix} -1.80(10)^{-4} & -.051 & 0 & 0 & 0 \\ 1.14(10)^{-4} & 4.66(10)^{-3} & 0 & -.032 & 0 \\ -4.44(10)^{-7} & .141 & 0 & -.029 & 0 \end{bmatrix} \qquad (4\text{-}35)$$

The uncompensated, model, and compensated system were simulated using CSMP as in Example 4.1. Figures 4-13 through 4-22 give the results of the simulation. Again note that the compensated system and the model time response are identical, indicating their transfer functions are equal.

From the tabulated values of N_α, $\omega_{n_{sp}}$, and ζ_{sp} the compensated aircraft peformance was plotted on graphs corresponding to Shomber and Gertsen's performance criteria. Figures 4-23 through 4-30 show the results. Each plot is at a fixed velocity while the altitude is varied. The plots indicate that the aircraft performs satisfactorily for all flight conditions plotted.

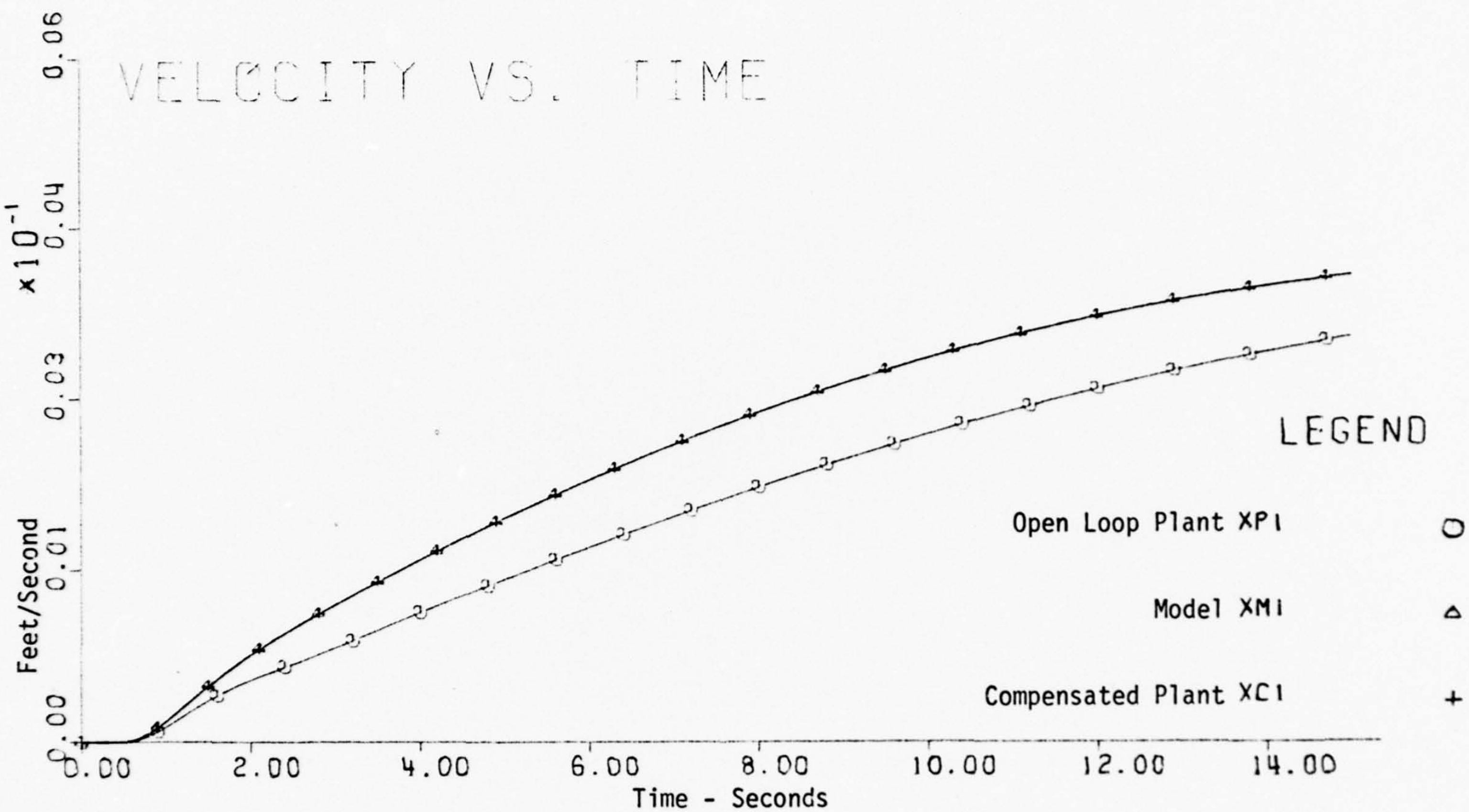

Figure 4-13. Change in Velocity vs. Time For System in Example 4.2.

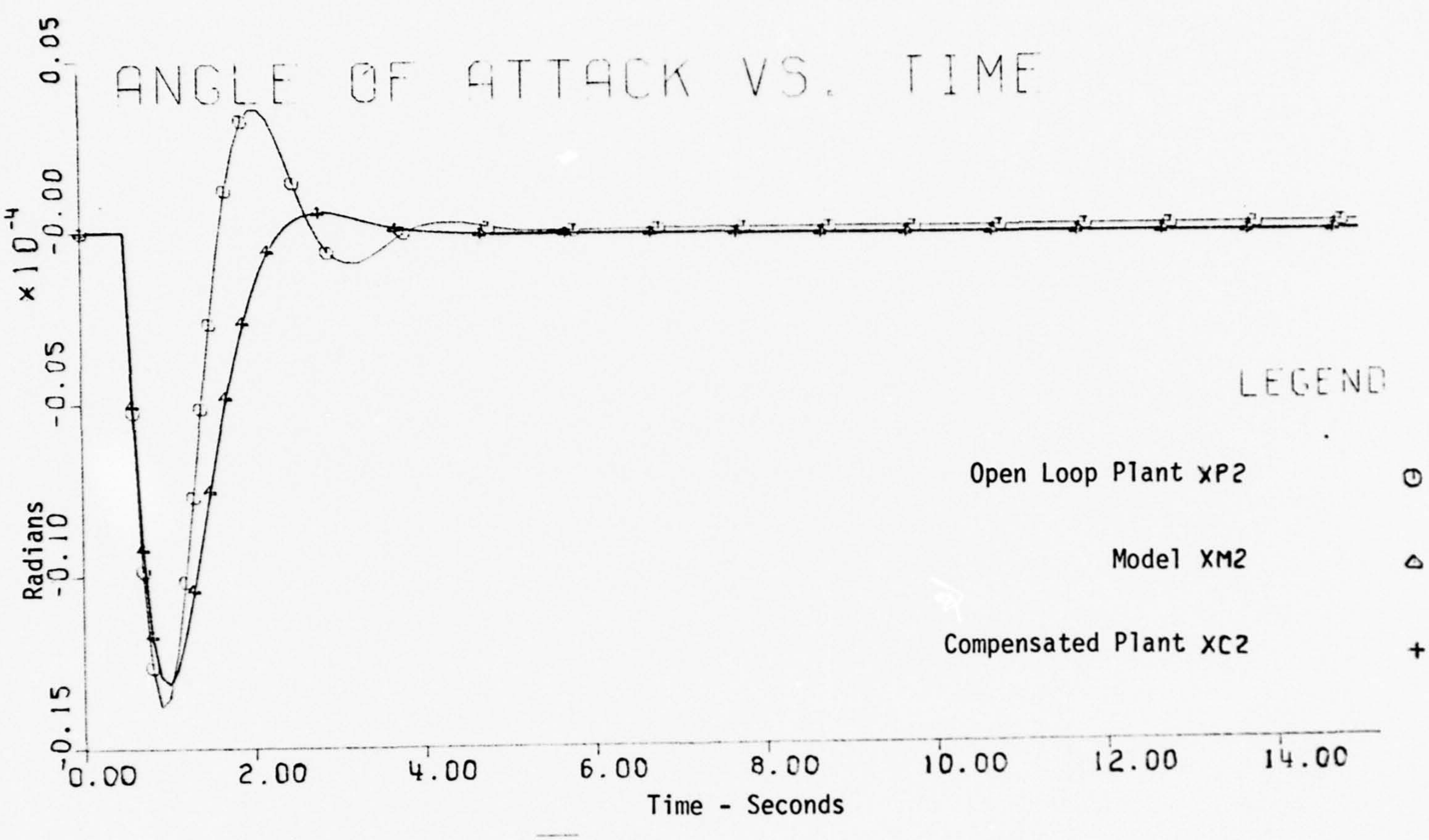

Figure 4-14. Change in Angle of Attack vs. Time For System in Example 4.2.

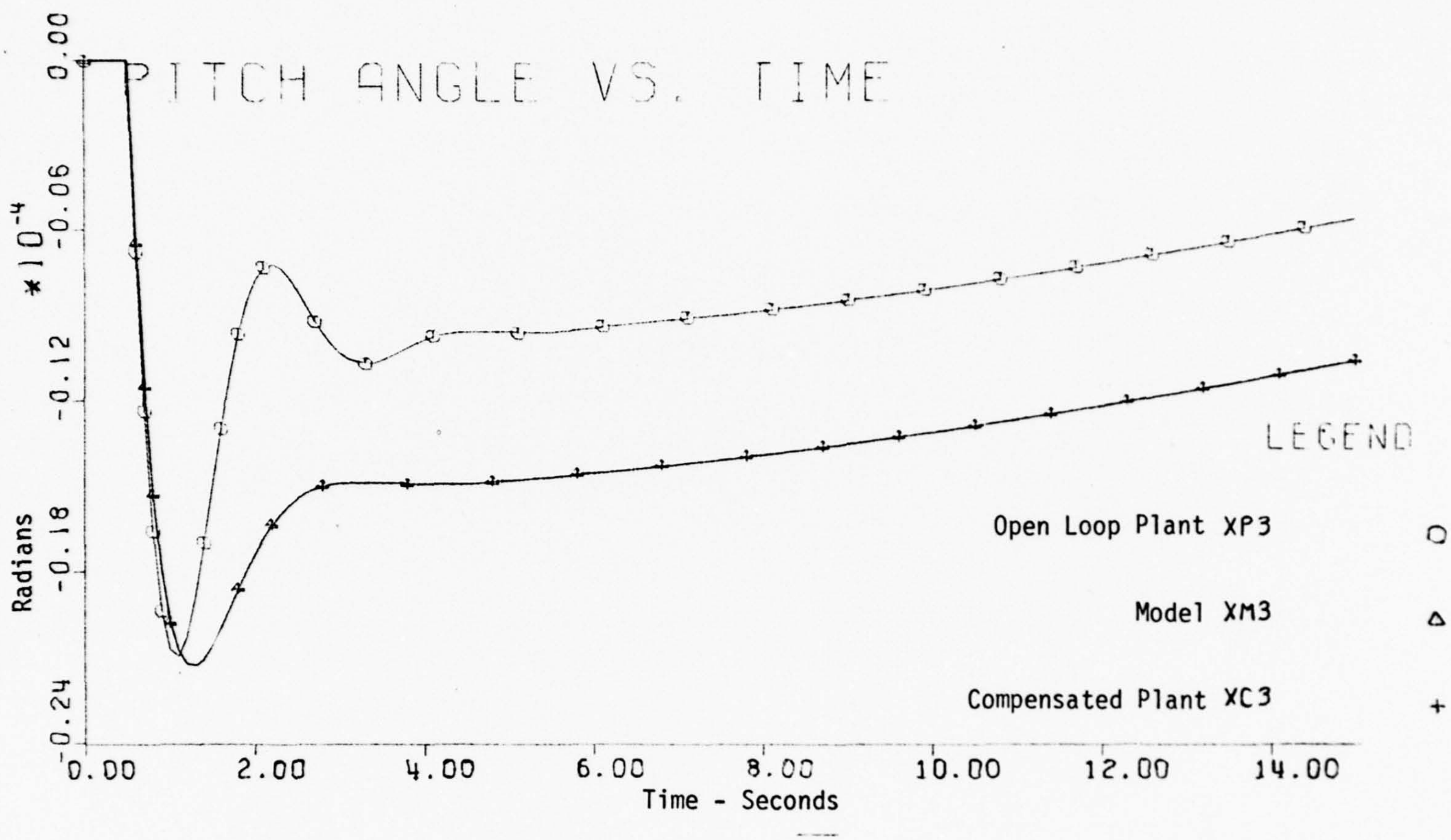

Figure 4-15. Change in Pitch Angle vs. Time for System in Example 4.2.

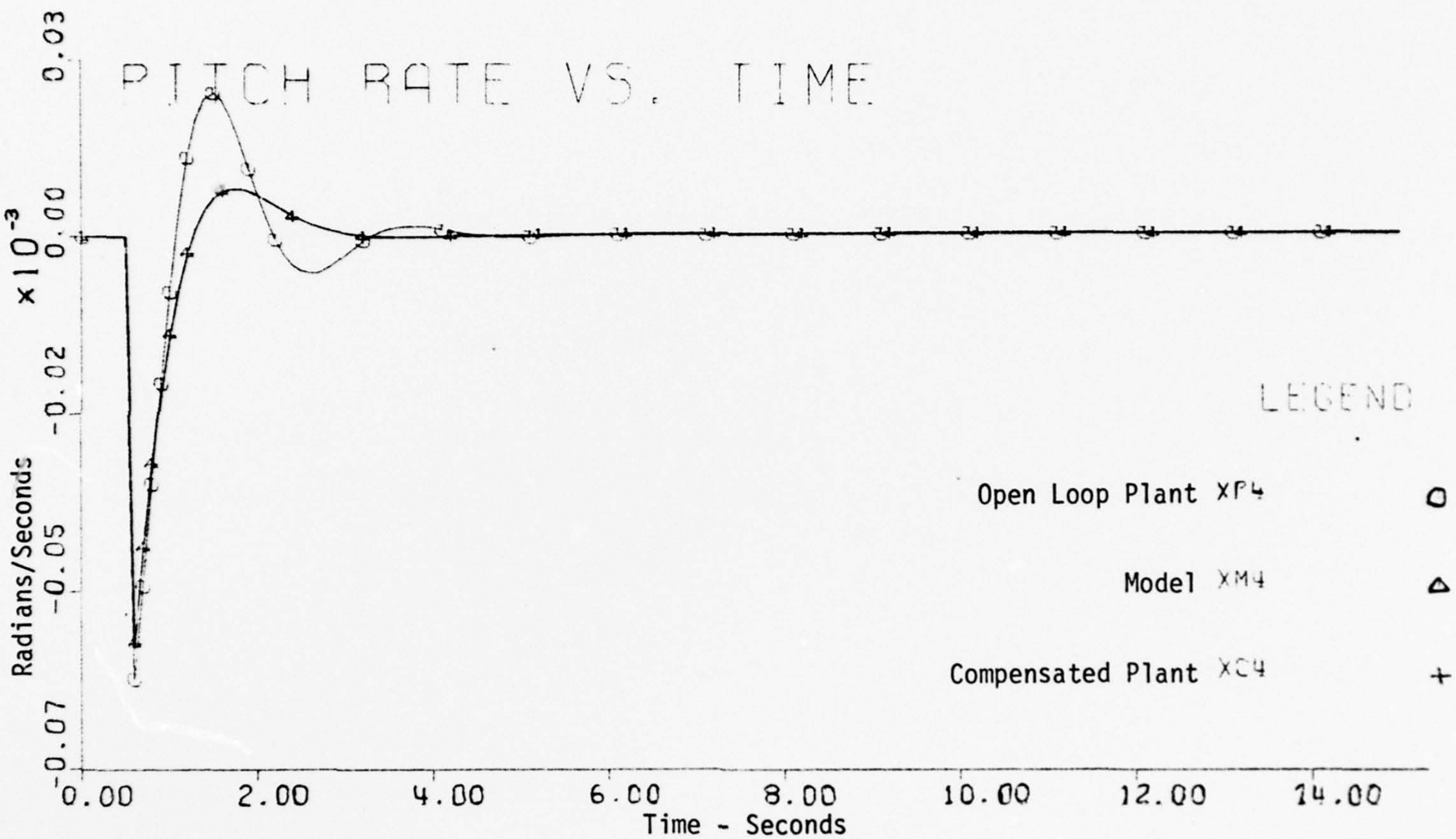

Figure 4-16. Pitch Rate vs. Time For System in Example 4.2.

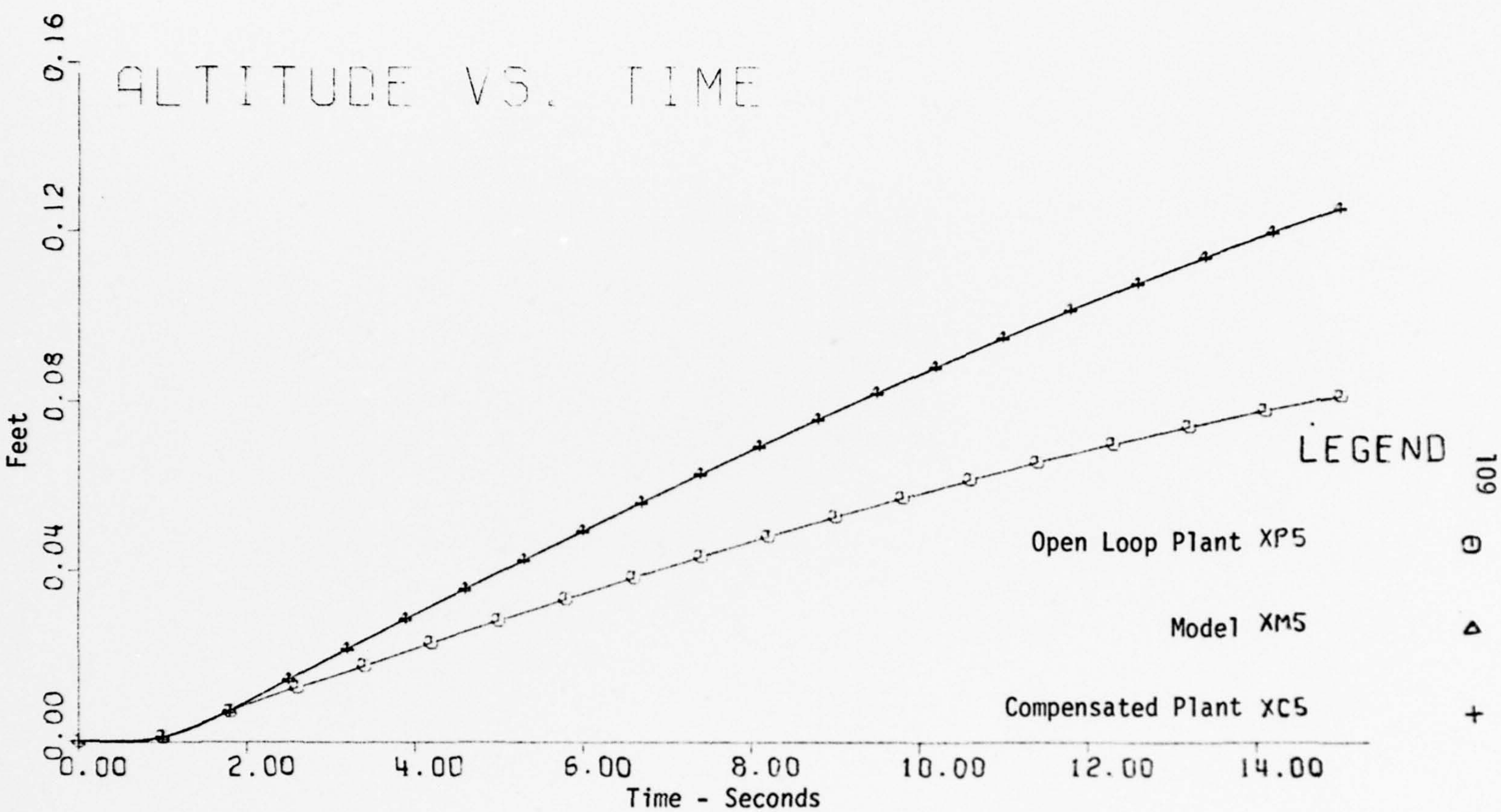

Figure 4-17. Change in Altitude vs. Time For System in Example 4.2.

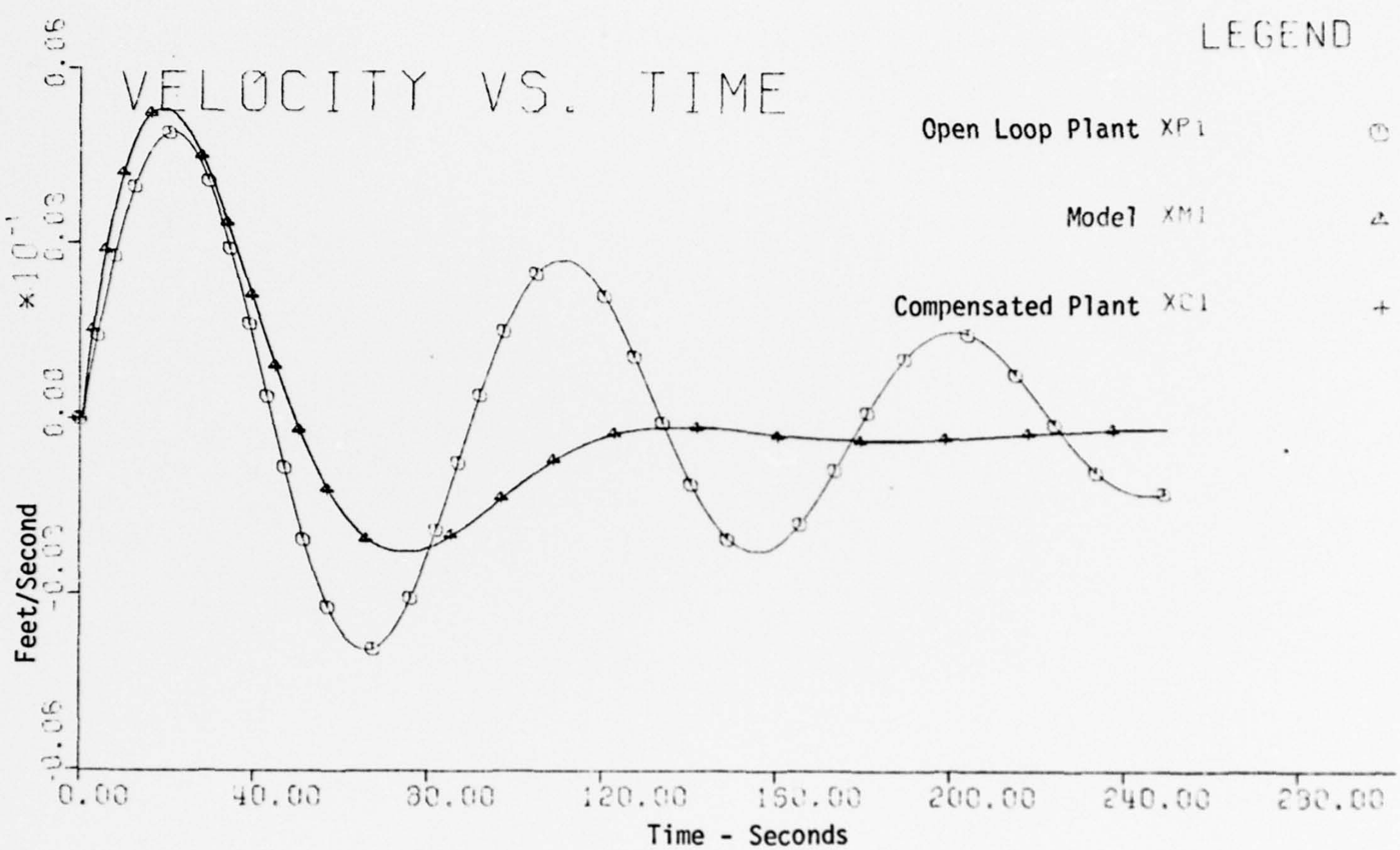

Figure 4-18. Change in Velocity vs. Time For System in Example 4.2.

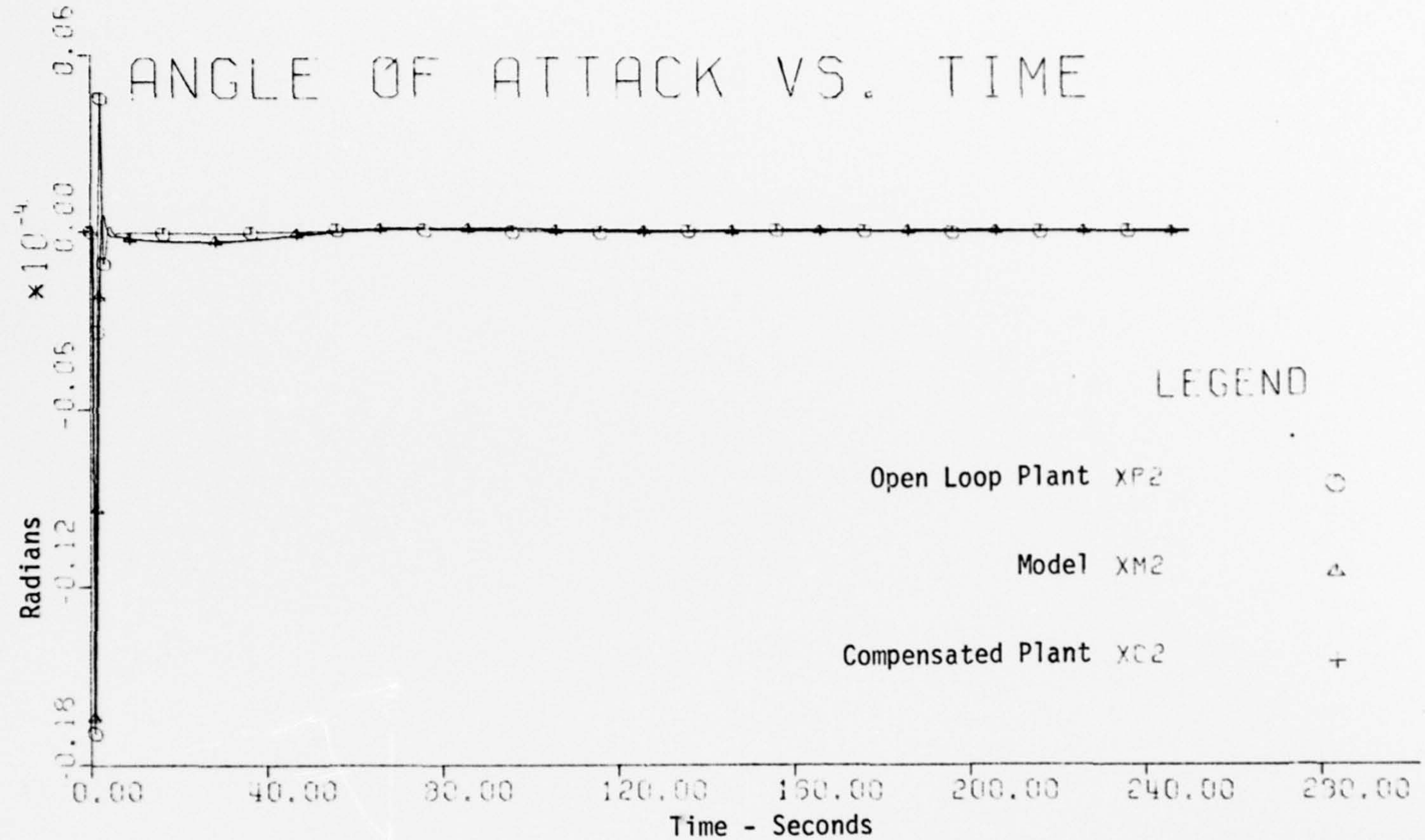

Figure 4-19. Change in Angle of Attack vs. Time For System Example 4.2.

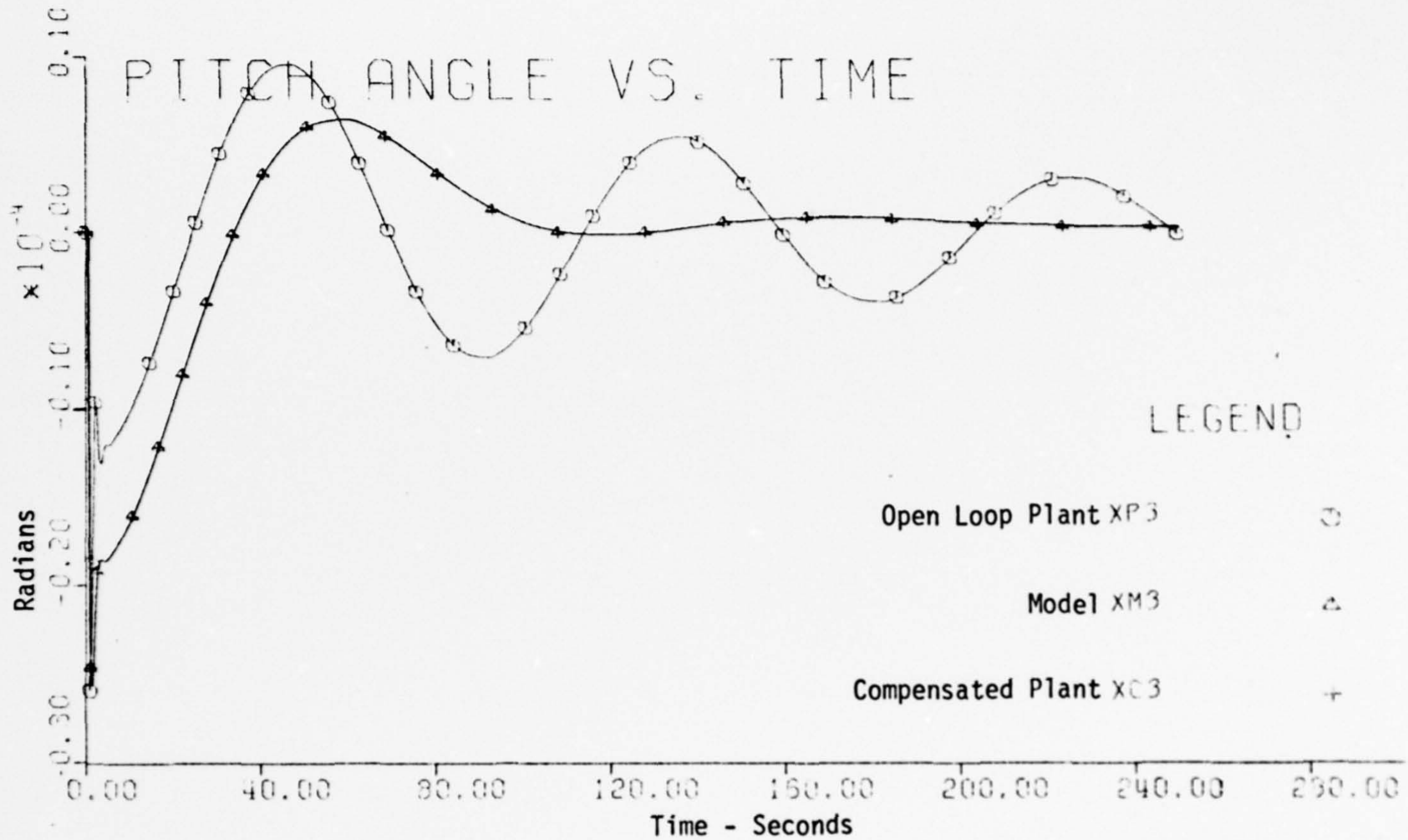

Figure 4-20. Change in Pitch Angle vs. Time For System in Example 4.2.

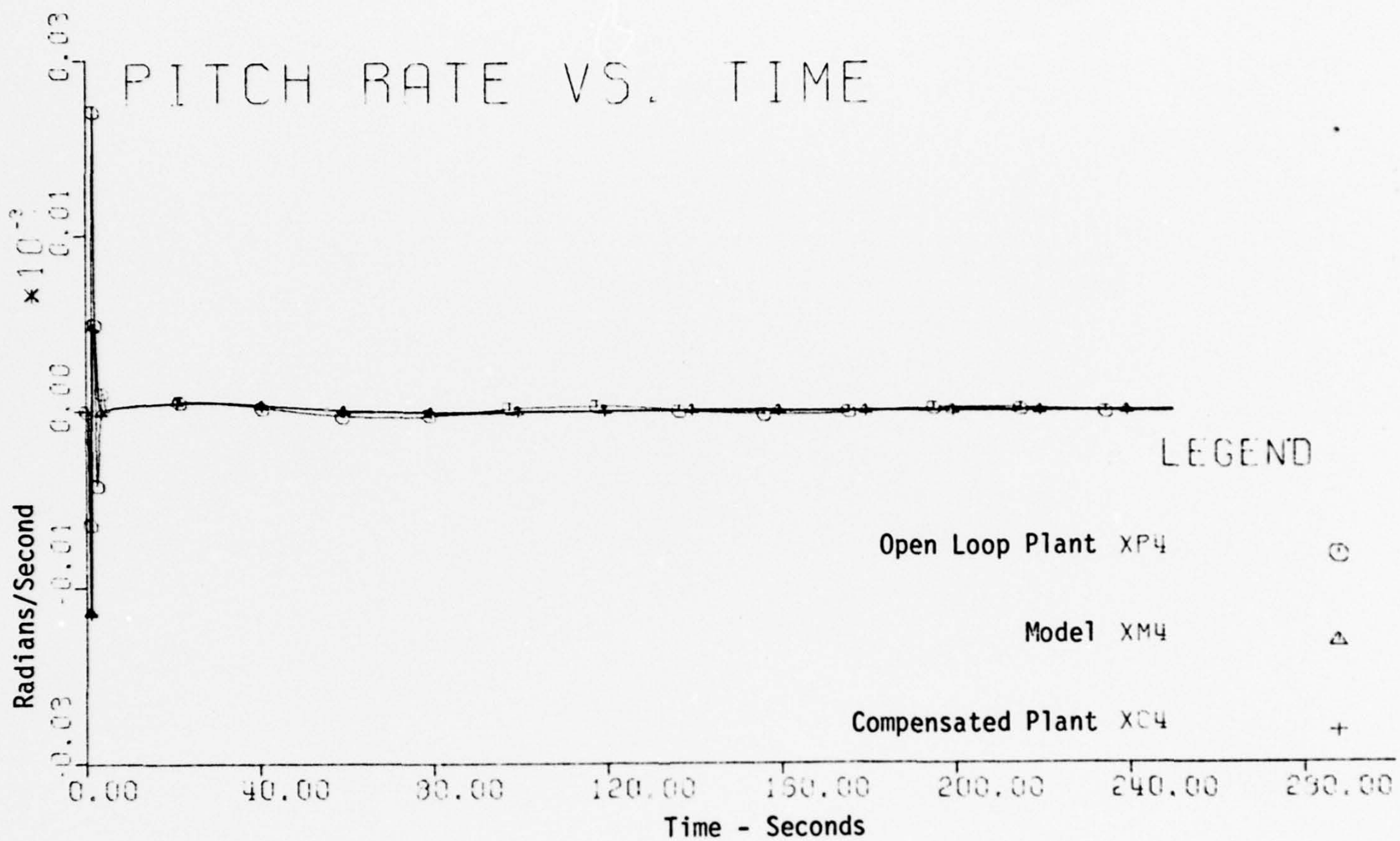

Figure 4-21. Pitch Rate vs. Time For System in Example 4.2.

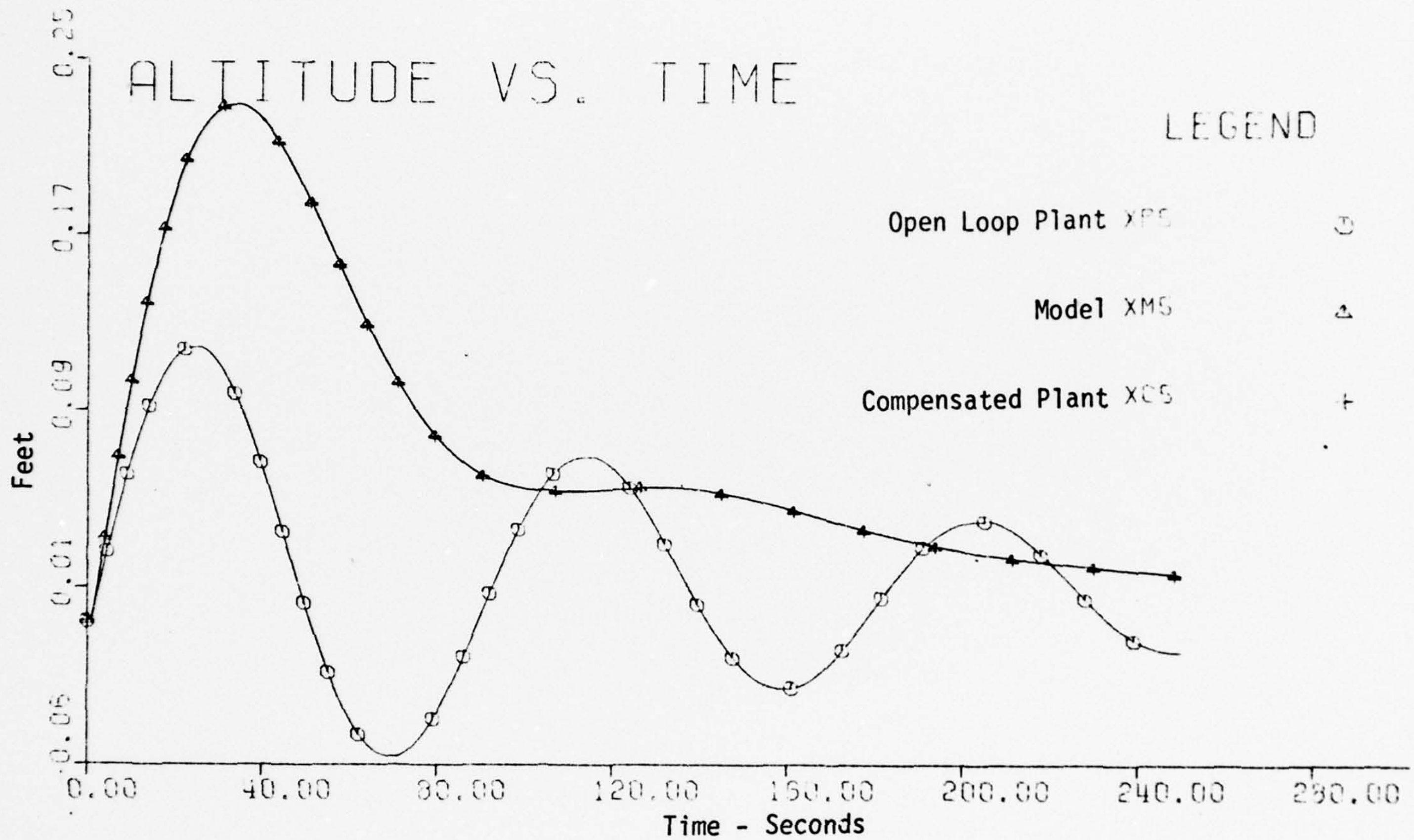

Figure 4-22. Change in Altitude vs. Time For System in Example 4.2.

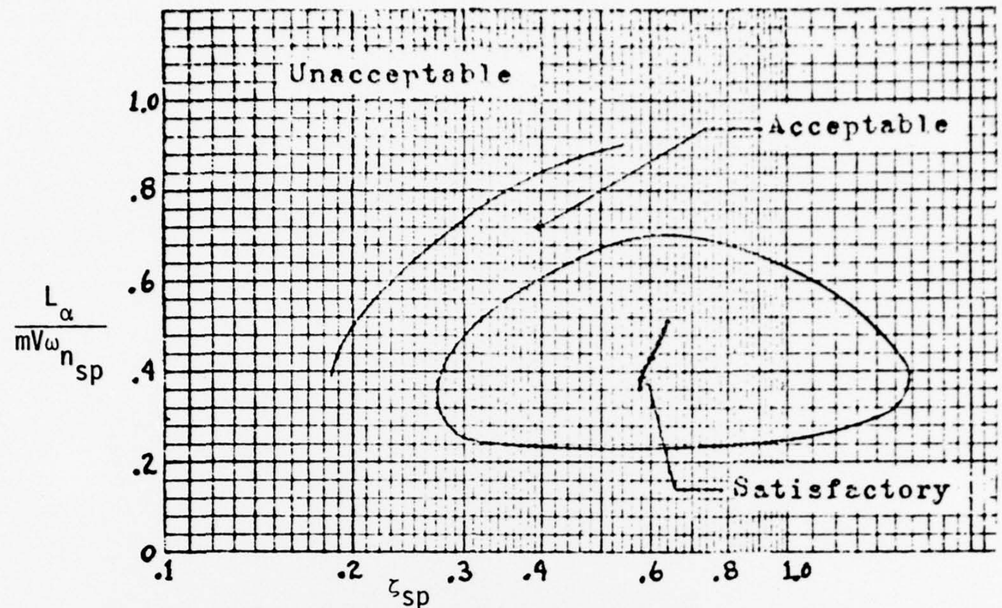

Figure 4-23. Handling Qualities of Compensated System In Example 4.2 for Velocity = 330 fps and Altitude Ranging From Sea Level to 30,000 ft.

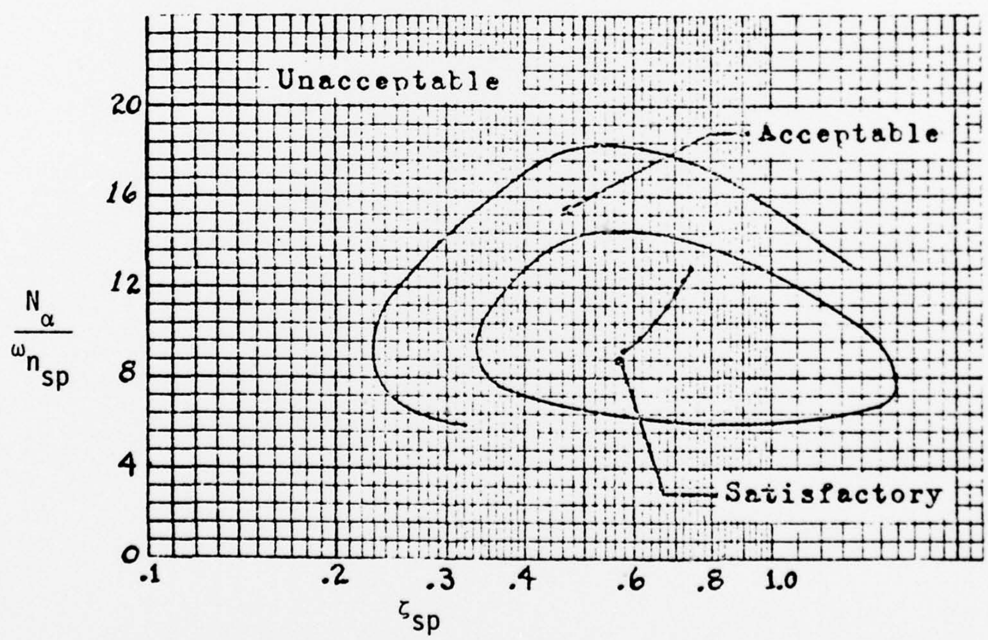

Figure 4-24. Handling Qualities of Compensated System In Example 4.2 for Velocity = 880 fps and Altitude Ranging From Sea Level to 30,000 ft.

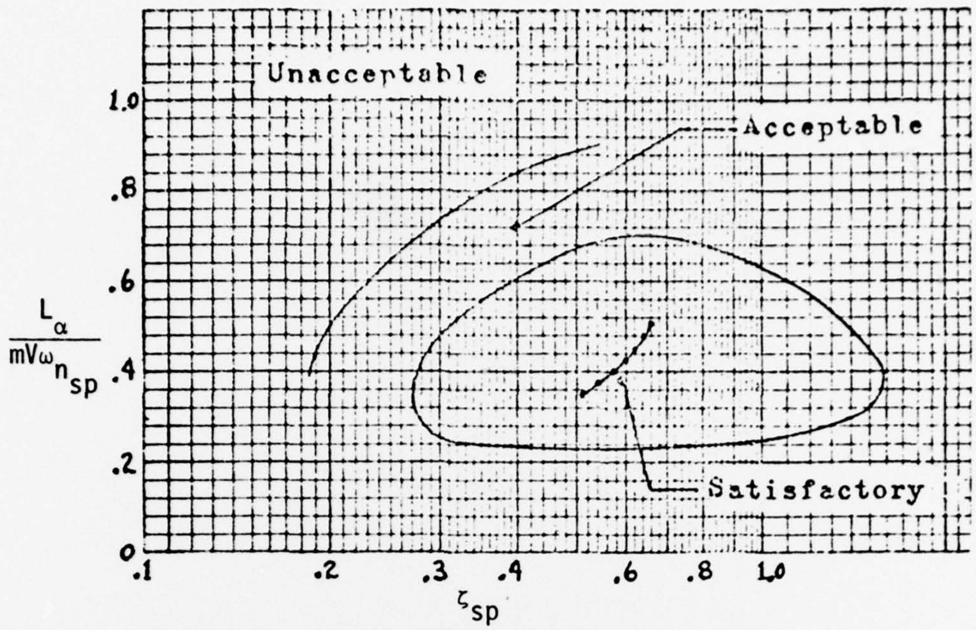

Figure 4-25. Handling Qualities of Compensated System In Example 4.2 for Velocity =440 fps and Altitude Ranging From Sea Level To 30,000 ft.

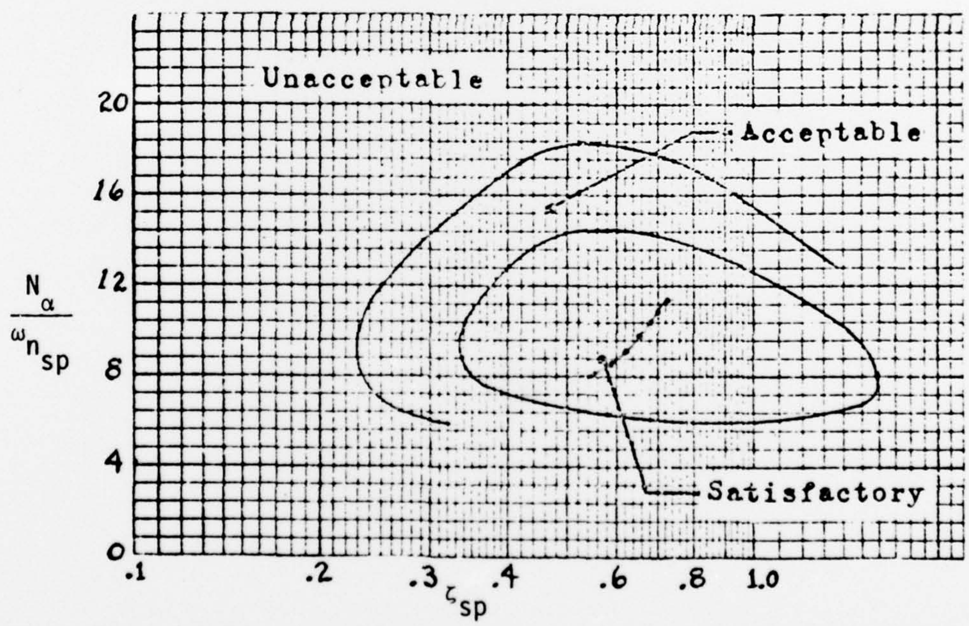

Figure 4-26. Handling Qualities of Compensated System In Example 4.2 For Velocity = 770 fps and Altitude Ranging From Sea Level To 30,000 ft.

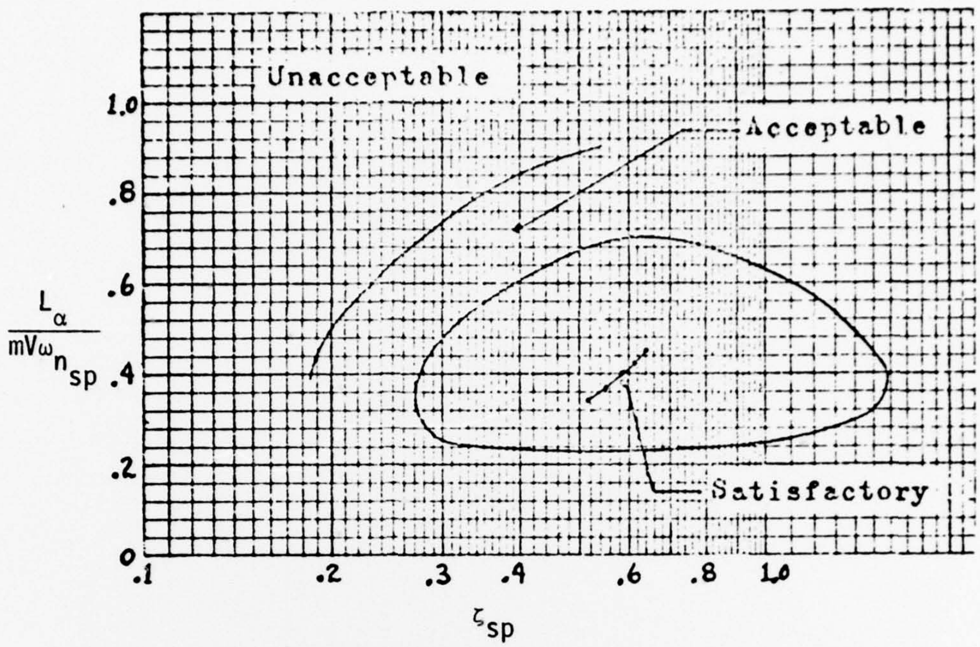

Figure 4-27. Handling Qualities of Compensated System In Example 4.2 for Velocity = 550 fps and Altitude Ranging From 10,000 ft. to 30,000 ft.

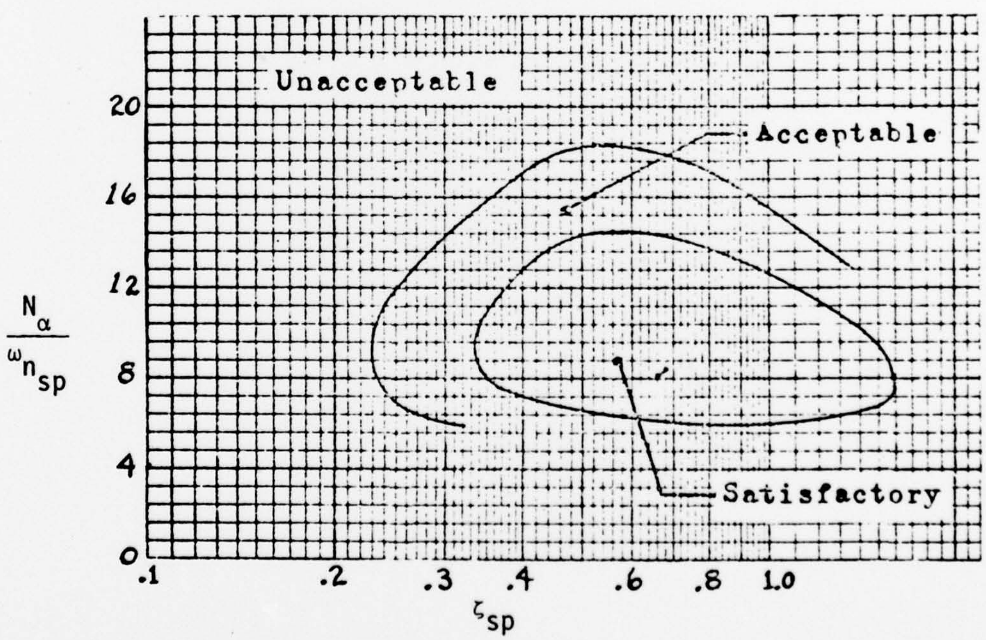

Figure 4-28. Handling Qualities of Compensated System in Example 4.2 For Velocity = 550 fps and Altitude Ranging From Sea Level to 5,000 ft.

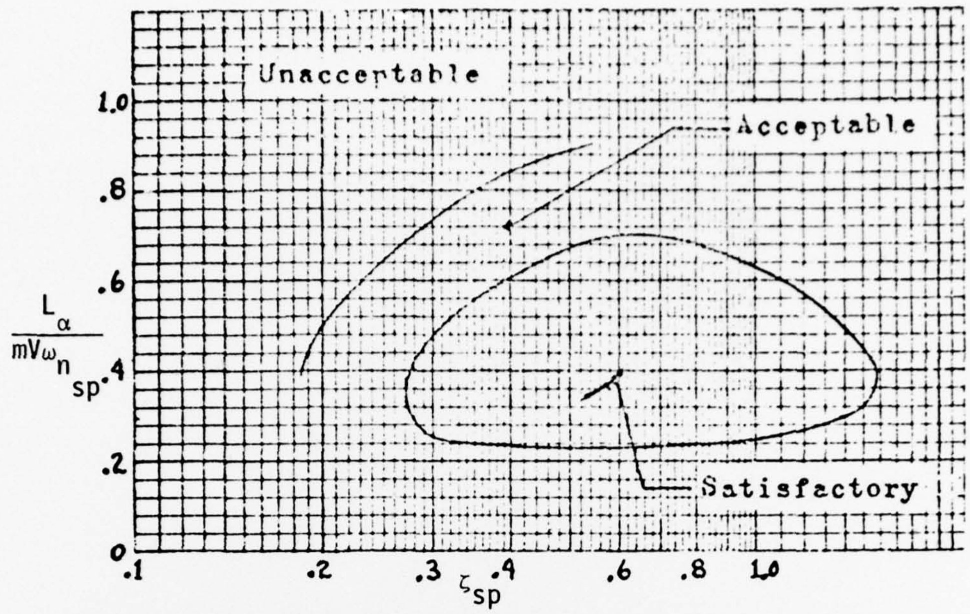

Figure 4-29. Handling Qualities of Compensated System in Example 4.2 for Velocity = 660 fps and Altitude Ranging From 20,000 ft. to 30,000 ft.

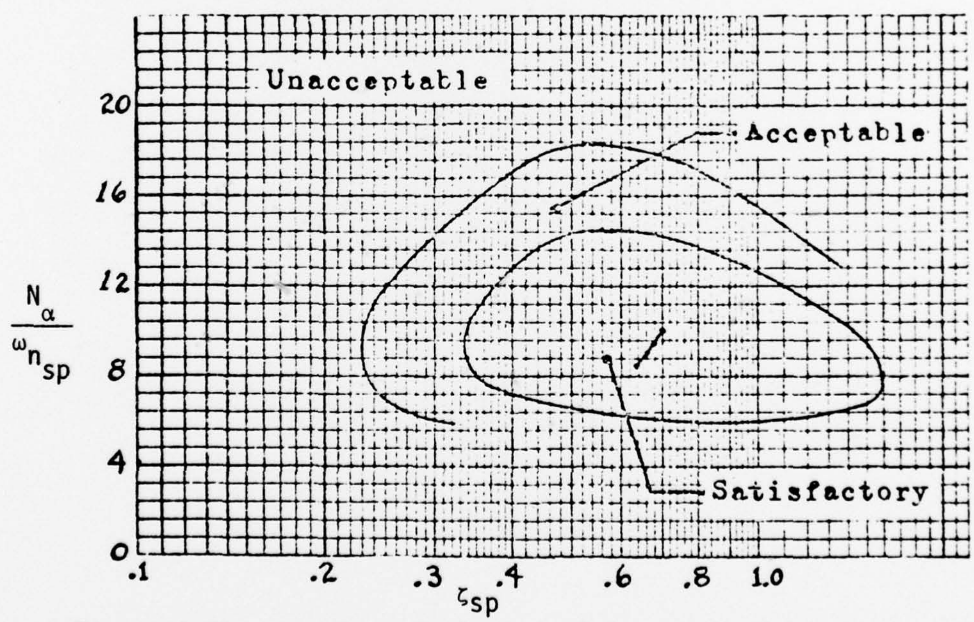

Figure 4-30. Handling Qualities of Compensated System in Example 4.2 for Velocity = 660 fps and Altitude Ranging from Sea Level to 15,000 t.

V. SUMMARY AND CONCLUSIONS

The longitudinal aircraft equations were investigated to determine which elements of the A matrix dominate the response of the aircraft. A method for contriving a model that has a desired response based on performance criteria was developed. Two methods for designing a compensator which forces the aircraft to respond in the same manner as the desired model were presented.

The method for designing the model is straightforward. The system's short period response is easily and exactly adjusted by altering three elements of the A matrix. The phugoid response was harder to adjust precisely because of the indefiniteness of the dominant phugoid elements in the A matrix. This resulted in a trial and error method for obtaining the exact desired phugoid response. However, it should be noted that this difficulty in adjusting the phugoid response is not a major problem since the phugoid criterion is not precise.

A special case based on equality of the plant and model similarity transformations in Curran's algorithm was presented. The criteria for equality of the similarity transformations were derived and it was shown that the aircraft equations in state space form met the criteria if the systems are controllable. The advantage of Curran's algorithm over other methods using optimal control theory is the directness of solution. The algorithm was easily implemented in a computer program

and a solution to the program involved a minimum of human/computer interaction.

The second method for designing the compensator was more direct than Curran's method since it did not employ any similarity transformations nor was it necessary to augment the system as in Curran's method. The alternate method imposed the same restrictions on the system matrices as Curran's method, e.g. the B matrix must have full rank and the model must be structurally similar to the plant. It was shown that a necessary condition for the similarity transformations to be equal in Curran's method was that the model and plant input matrices had to be equal. The alternate method did not impose this restriction. There were restrictions on the alternate method involving the structure of the model matrices and the plant input matrix. They are presented in Proposition 3 in Chapter III.

The restriction that the input matrix have full rank was shown to be no limitation for the compensator design for any controllable system. In Proposition 2, it was shown that the input matrix can be made full rank by not utilizing all of the control inputs.

In Chapter IV, compensators were designed for two aircraft using both Curran's method and the alternate method. The results were more than satisfactory. Simulation results verified that the compensated system response was equal to the model response. It was also shown in Example 4.2 that, once the aircraft is compensated so that its response at some arbitrary median flight condition is well within the satisfactory region of the performance criteria plots, then the system

response will be satisfactory over a range of flight conditions. Example 4.2 verified the derivation given in Equations (2-34) through (2-43) for determining how the compensated system ζ_{sp} and $\omega_{n_{sp}}$ varied with velocity and altitude. The fact that the compensated system response is satisfactory for a range of flight conditions suggests that, for aircraft that have a large flight envelope such as supersonic or high altitude aircraft, several compensators could be designed, each for overlapping flight regimes. This would involve some sort of sensing and switching control circuitry to select the proper compensator for the corresponding flight regime.

REFERENCES

[1] J. Roskam, Flight Dynamics of Rigid and Elastic Airplanes. Lawrence, Kansas: The University of Kansas, 1973.

[2] E. G. Rynaski, P. A. Reynolds, and W. A. Shet, Design of Linear Flight Control Systems Using Optimal Control Theory. Wright-Patterson A.F.B. Technical Report ASD-TDR-63-376.

[3] C. A. Windsor and R. J. Roy, "The Application of Specific Optimal Control to the Design of Desensitized Model Following Control Systems," IEEE Transactions on Automatic Control, Vol. AC-15, No. 3, pp. 326-333, June 1970.

[4] R. T. Curran, "Equicontrollability and the Model Following Problem," Technical Report TR-6302-2, Information Systems Laboratory, Stanford University, July 1971.

[5] H. A. Shomber and W. M. Gertsen, "Longitudinal Handling Qualities Criteria: An Evaluation," Journal of Aircraft, Vol. 4, No. 4, pp. 371-376, July-August 1967.

[6] B. Etkin, Dynamics of Atmospheric Flight. New York: John Wiley and Sons, Inc., 1972.

[7] D. G. Luenberger, "Canonical Forms for Linear Multivariable Systems," IEEE Transactions on Automatic Control, Vol. AC-12, No. 3, pp. 290-293, June 1967.

[8] D. O. Anderson and D. G. Luenberger, "Design of Multivariable Feedback Systems," Proceedings of IEE , Vol. 114, No. 3, pp. 395-399, March 1967.

[9] I. D. Landau and B. Cortiol, "Adaptive Model Following Systems for Flight Control and Simulation," Journal of Aircraft, Vol. 9, No. 9, pp. 668-674, 1972.

[10] P. Murdoch, "Pole and Zero Assignment by Proportion Feedback," IEEE Transactions on Automatic Control, October 1973.

[11] Flying Qualities of Piloted Aircraft, Military Specification, MIL-F-8785B(ASG), August 1969.